Published by Institute of Physics Publishing for SISSA

Received: *September 27, 2007*
Accepted: *November 5, 2007*
Published: *November 26, 2007*

Branes, anti-branes and Brauer algebras in gauge-gravity duality

Yusuke Kimura and Sanjaye Ramgoolam

Department of Physics, Queen Mary, University of London,
Mile End Road, London E1 4NS U.K.
E-mail: y.kimura@qmul.ac.uk, s.ramgoolam@qmul.ac.uk

Abstract: We propose gauge theory operators built using a complex Matrix scalar which are dual to brane-anti-brane systems in $AdS_5 \times S^5$, in the zero coupling limit of the dual Yang-Mills. The branes involved are half-BPS giant gravitons. The proposed operators dual to giant-anti-giant configurations satisfy the appropriate orthogonality properties. Projection operators in Brauer algebras are used to construct the relevant multi-trace Matrix operators. These are related to the "coupled representations" which appear in 2D Yang-Mills theory. We discuss the implications of these results for the quantum mechanics of a complex matrix model, the counting of non-supersymmetric operators and the physics of brane-anti-brane systems. The stringy exclusion principle known from the properties of half-BPS giant gravitons, has a new incarnation in this context. It involves a qualitative change in the map between brane-anti-brane states to gauge theory operators. In the case of a pair of sphere giant and anti-giant this change occurs when the sum of the magnitudes of their angular momenta reaches N.

Keywords: AdS-CFT Correspondence, D-branes, Gauge-gravity correspondence.

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1. Introduction

The two-point functions of gauge theory operators in $N = 4$ U(N) super-Yang-Mills gauge theory corresponding to highest weights of half-BPS representations can be diagonalised [1]. The elements of the diagonal basis are given in terms of $\chi_R(\Phi)$ where R is a Young Diagram of n boxes

$$\chi_R(\Phi) = \frac{1}{n!} \sum_{\sigma \in S_n} \chi_R(\sigma) tr(\sigma \Phi) \tag{1.1}$$

$\chi_R(\sigma)$ is the character of σ in the representation of S_n labelled by R. Φ is a complex matrix which can be viewed as an operator acting on an N-dimensional vector space V, i.e $\Phi : V \to V$. It can be extended to give an operator transforming $V^{\otimes n} \to V^{\otimes n}$ by considering $\Phi \otimes \Phi \cdots \otimes \Phi$. In the r.h.s. of (1.1) the trace is being taken in $V^{\otimes n}$, and σ acts on $V^{\otimes n}$ by permuting the factors. From these facts it follows that

$$tr(\sigma\Phi) = \Phi^{i_1}_{i_{\sigma(1)}} \cdots \Phi^{i_n}_{i_{\sigma(n)}} \tag{1.2}$$

The operator $\chi_R(\Phi)$ can also be viewed as a holomorphic continuation of the U(N) character $\chi_R(U)$ by replacing the unitary matrix U with a complex matrix Φ. It is also useful to view it as a trace $tr_n(\frac{p_R}{d_R}\Phi)$ in $V^{\otimes n}$ obtained by using a projection operator p_R in the group algebra of S_n

$$p_R = \frac{d_R}{n!} \sum_{\sigma \in S_n} \chi_R(\sigma)\sigma \tag{1.3}$$

The 2-point function in this basis of operators is diagonal

$$< \chi_R(\Phi^\dagger(x_1))\chi_S(\Phi(x_2)) >= \frac{\delta_{RS} f_R}{(x_1 - x_2)^{2n}} \tag{1.4}$$

where f_R is a simple group theoretic quantity. This is derived using the basic formula

$$< \Phi^{\dagger}{}_j^{\ i}(x_1)\Phi^k_l(x_2) >= \frac{\delta^i_l \delta^j_k}{(x_1 - x_2)^2} \tag{1.5}$$

In most of this paper we will not be interested in the position dependences, so we will drop the x's. Having a diagonal basis in the space of half-BPS operators allows an identification of gauge theory operators corresponding to half-BPS giant gravitons [2−4] in $AdS_5 \times S^5$ space-time via the AdS/CFT duality [5−7]. Some further aspects and developments related to half-BPS giant gravitons are in [1, 8−16] and references therein.

For an appropriate choice of R, $\chi_R(\Phi)$ is dual to a sphere-giant graviton, which is a spherical three-brane moving in S^5. As we will explain, by replacing the 3-brane with an anti-3-brane, and at the same time reversing the direction of rotation, we also have a solution of the same energy. This anti-giant is dual to $\chi_R(\Phi^\dagger)$. The same remark applies to AdS-giants (also known as dual giants). The main interest in this paper is to investigate candidates for systems of giant and anti-giants. This requires a diagonalisation of the two-point function in the space of operators built from both Φ and $\Phi^\dagger$. This problem can be solved elegantly in terms of *Brauer algebras*. These algebras are parametrised by two positive integers. For the case of m copies of Φ and n copies of $\Phi^\dagger$ the relevant algebra is $B_N(m,n)$. The associative algebra $B_N(m,n)$ contains the group algebra of the product of symmetric groups $S_m \times S_n$, which is denoted as $\mathbb{C}[S_m \times S_n]$.

An outline of the proposal for gauge-theory duals of giant-anti-giants and the role of Brauer algebras will be given in section 2. These Brauer algebras will be introduced more systematically, their relevance and useful properties explained in section 3. Of particular interest in constructing duals of brane-anti-brane composites will be a subset of the *orthogonal central projectors* in the Brauer algebra. By central, we mean that the projectors commute with the Brauer algebra. Section 4 will be devoted to techniques for the explicit construction of these orthogonal projectors. Examples of these projectors will be given in section 5.

The gauge invariant operators constructed from central Brauer projectors do not exhaust the complete set of gauge invariant operators that can be constructed from Φ and $\Phi^\dagger$. To get the complete set we need to consider *symmetric Brauer elements*. The counting of the gauge invariant operators in the limit where the Matrices Φ is large is known to be given by Polya theory. In section 6, we relate the Polya counting to Brauer algebras. In section 7 we describe an orthogonal basis in the space of symmetric Brauer elements and show how they lead to a diagonal basis for the two-point functions of multi-trace operators. A physical interpretation in terms of brane-anti-branes of the orthogonal basis of multi-trace operators is discussed in section 8. This includes a discussion of an interesting finite N effect we describe as the *nonchiral stringy exclusion principle*.

The reader is not assumed to have any prior knowledge about Brauer algebras. A summary of useful results is given in section 3, along with references to the mathematical literature. For a reader with interest in Brauer algebras, we point out the new formula for dual Brauer elements (3.27). The explicit formulae for projectors in section 4 and 5 and the connection with Polya theory of sections 3.4 and 6 should also be of interest from this mathematical point of view. For the reader familiar with the large N expansion of two-dimensional Yang-Mills we would point to the new formula for coupled dimensions (4.18) as an appetiser. Explicit examples of orthogonal bases of multi-matrix operators are given in the appendices.

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2. Proposal for gauge theory duals of brane-anti-brane systems: outline

Giant 3-brane gravitons are dual to $\chi_R(\Phi)$. A simple inspection of the derivation of the giant graviton solutions of [2] shows that spherical anti-3-branes can provide supersymmetric solutions with opposite angular momentum. When we change the angular velocity $\dot\phi$ to $-\dot\phi$ while changing the sign of the Chern-Simons coupling to the background flux, as appropriate for changing brane to anti-brane, the effective Lagrangian and Hamiltonian are unchanged, while the angular momentum changes sign. This leads to the conclusion that anti-branes also provide supersymmetric solutions. Giant anti-3-brane gravitons are dual to $\chi_S(\Phi^\dagger)$. The brane-anti-brane composites will be non-supersymmetric.

Gauge theory operators dual to brane-anti-brane systems involve both Φ and $\Phi^\dagger$. In the free field limit, the construction of composite operators such as $\chi_R(\Phi)$ is simple. No short distance subtractions are required, due to the vanishing two point function

$$< \Phi(x)\Phi(y) >= 0 \tag{2.1}$$

If we wish to consider a local operator of the form $tr(\Phi)tr(\Phi^\dagger)$ we have to subtract a short distance singularity. Denoting the well-defined local operator as $: tr(\Phi(x))tr(\Phi^\dagger(x)) :$ we have

$$: tr(\Phi(x))tr(\Phi^\dagger(x)) : \;\; = Lim_{\epsilon\to 0} \;\; tr(\Phi(x)) \; tr(\Phi^\dagger(x+\epsilon)) - \frac{N}{\epsilon^2} \tag{2.2}$$

Note that the renormalised operator leads to a well-defined state

$$: tr(\Phi(x))tr(\Phi^\dagger(x)) : |0>$$

For example in computing the overlap of this state with $< 0|$ we get a well-defined correlator. Without the subtraction we would get a divergent answer for this overlap.

A naive guess for the gauge theory dual of a brane-anti-brane system would be $:\chi_R(\Phi)\chi_S(\Phi^\dagger):$. Such an operator is not in general orthogonal to operators which are of the form $: tr(\Phi\Phi^\dagger)\chi_{R_1}(\Phi)\chi_{S_1}(\Phi^\dagger) :$. In the simplest case of $m = 1, n = 1$, for example, $: tr(\Phi)tr(\Phi^\dagger) :$ is not orthogonal to $: tr(\Phi\Phi^\dagger) :$. However consider, in this case, an operator

$$\mathcal{O} = tr(\Phi)tr(\Phi^\dagger) - \frac{1}{N}tr(\Phi\Phi^\dagger) \tag{2.3}$$

This has a number of interesting and easily verified properties. The first is that its short distance subtractions vanish

$$: \mathcal{O} := \mathcal{O} \tag{2.4}$$

The second is that

$$< \mathcal{O} : tr(\Phi\Phi^\dagger) :>= 0 \tag{2.5}$$

In terms of the Young diagram classification of operators, $tr(\Phi)$ corresponds to the single box Young diagram denoted as [1]. So the operator $\mathcal{O}$ above can be viewed as a singularity free operator associated to the pair of Young diagrams $([1],[1])$, which is orthogonal to operators where $\Phi, \Phi^\dagger$ are in the same trace. Brauer algebras will allow us to associate such a singularity free operator to any pair of Young diagrams R, S in the large N limit. More precisely we will need $c_1(R)+c_1(S) \leq N$, where c_1 denotes the length of the first column of the Young diagram. The origin of this condition in representation theory comes from (3.8). When we consider a brane R and antibrane S with $c_1(R)+c_1(S) > N$, their composite is best viewed as an excited state of another pair of branes satisfying the bound. A related fact is that the naive guess : $\chi_R(\Phi)\chi_S(\Phi^\dagger)$: becomes completely dependent on operators where $\Phi, \Phi^\dagger$ appear in the same trace. This will be explained in section 8.

The Brauer technology is also useful in classifying operators in a zero-dimensional or one-dimensional Matrix Model. In the one-dimensional case it has been shown [1] (see also [13, 41, 42]) that the reduction of the four-dimensional action on $S^3 \times R$ leads to the Hamiltonian and SO(2) symmetry generator

$$\begin{aligned} H &= tr(A^\dagger A + B^\dagger B) \\ J &= tr(A^\dagger A - B^\dagger B) \end{aligned} \tag{2.6}$$

The construction of operators corresponding to giant gravitons involves gauge invariant states obtained by acting with $A^\dagger$ on the vacuum. For anti-giant gravitons, we act with $B^\dagger$ only. For systems consisting of composites of giant and anti-giant we act with both $A^\dagger$ and $B^\dagger$. We will find a diagonal basis in the space of such operators using the Brauer algebra. The two-point function (1.5), with its position dependence removed, also appears in the zero-dimensional complex matrix model introduced in [17] and studied more recently in [18 − 20], which has a partition function

$$Z = \int [d\Phi d\Phi^\dagger] e^{-tr(\Phi\Phi^\dagger)} \tag{2.7}$$

Hence our results are also relevant to this zero-dimensional matrix model. Finally we expect that the results on projectors should help in a better understanding of the stringy interpretation of the non-chiral large N expansion of intersecting Wilson loops in two-dimensional Yang-Mills (2dYM). We will be making contact with some of the character and dimension formulae which play a role in the non-chiral expansion of 2dYM [21 − 23].

Having given away what Brauer algebras do for us, it is time to describe them more precisely and to explain why they are useful in understanding the properties of large composite operators in gauge theory.

3. Gauge invariant operators, correlators and brauer algebras

In understanding the role of Brauer algebras in the calculation of correlation functions of operators, it is useful to express the basic formulae from field theory in a diagrammatic notation for operators acting on V or $V^{\otimes n}$ or more generally on $V^{\otimes m} \otimes V^{*\otimes n}$. The use of diagrams for representing operators on tensor space plays a crucial role in knot theory [24] and has also been developed by physicists [25]. In [26] the tensor space diagrams are used

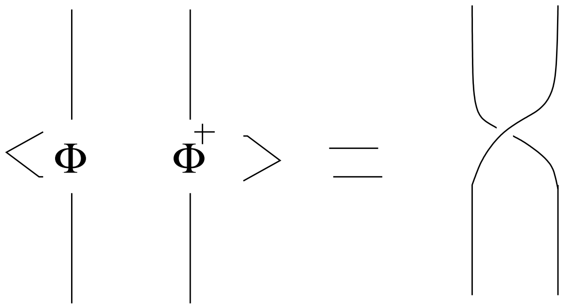

Figure 1: Correlator of Φ and $\Phi^\dagger$ as permutation operator.

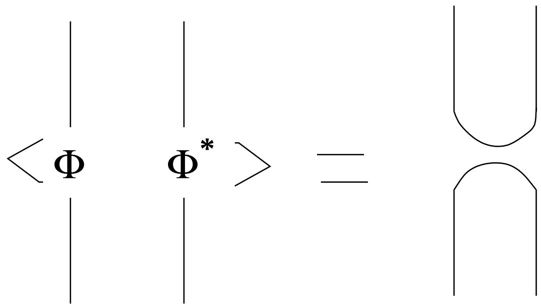

Figure 2: Correlator of Φ and Φ^* as contraction operator.

in the calculation of Wilson loops in two-dimensional Yang-Mills theory. In section 3 of [9] some of these diagrammatic techniques were summarised and used to simplify proofs of properties of correlators of large dimension multi-traces [1].

3.1 Correlators and brauer algebras

Consider the basic 2-point function obtained by doing the free-field path integral over Matrices

$$\langle \Phi^i_j \Phi^{\dagger k}_l \rangle = \delta^i_l \delta^k_j \tag{3.1}$$

where we have dropped the spacetime-dependence. By recognising Φ as an operator on V and σ as an operator on $V \otimes V$ with matrix elements

$$(\sigma)^{ik}_{jl} = \delta^i_l \delta^k_j \tag{3.2}$$

we can re-write (3.1) in an index-free form as

$$\langle \Phi \otimes \Phi^\dagger \rangle = \sigma \tag{3.3}$$

This can be expressed in a precise diagrammatic form in figure 1. The permutation on the r.h.s. is a map from $V \otimes V$ to $V \otimes V$. The power of the diagrammatic presentation comes from the fact that essentially the same diagram represents the 2-point function when we

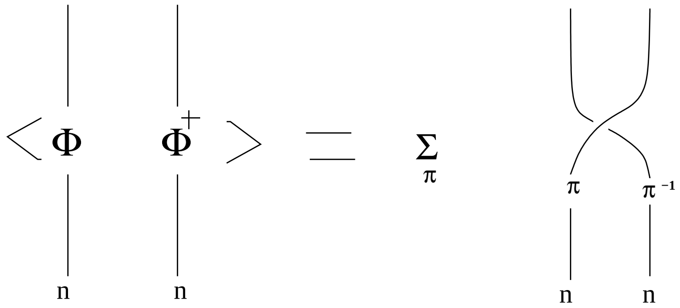

Figure 3: Correlator of n copies of Φ and n copies of $\Phi^\dagger$.

have n copies of Φ and n copies of $\Phi^\dagger$. Now we have Φ as an operator on $V^{\otimes n}$ which is simply denoted by a line labelled by n. Φ is understood to act on this as $\Phi\otimes\Phi\otimes\cdots\otimes\Phi$. The result of the correlator is to have the same twist, but in addition, a sum over permutations in S_n (denoted by π in figure 3) which determines which of the n Φ's is contracted with which of the n $\Phi^\dagger$ (for uses of this diagrammatic formula see [9]).

Rather than writing the 2-point function in terms of Φ and $\Phi^\dagger$ we can use the complex conjugate Φ^* to write

$$\langle \Phi^i_j \Phi^{*k}_l \rangle = \delta^{ik}\delta_{jl} \tag{3.4}$$

We can view Φ^* as a map from conjugate $\bar{V}$ to $\bar{V}$ and $\Phi\otimes\Phi^*$ as a map from $V\otimes\bar{V}$ to $V\otimes\bar{V}$. To describe the right-hand side of (3.4) in algebraic terms we introduce the linear operator C as a map from $V\otimes\bar{V}$ to $V\otimes\bar{V}$

$$(C)^{ik}_{jl} = \delta^{ik}\delta_{jl} \tag{3.5}$$

We can now write the index free form of (3.4) as

$$\langle \Phi\otimes\Phi^* \rangle = C \tag{3.6}$$

This can be expressed in a precise diagrammatic form in figure 2.

Note that in (3.4) the indices k,l are viewed as labelling vectors in $\bar{V}$ whereas the (i,j) denote vectors in V. We can make that explicit in the diagrams by introducing arrows, but this is a refinement of the diagrammatic notation, which is not crucial, though it is sometimes useful in manipulating the diagrams.

3.2 Brauer algebra: definition and Schur-Weyl duality

We review known facts about Brauer algebras mostly from [27]. Other useful references are [28 – 33].

Recall that S_n is the centraliser of $\mathrm{U}(N)$ (or $GL(N)$) acting on $V^{\otimes m}$. Hence we have Schur-Weyl duality

$$V^{\otimes n} = \oplus_R V_R^{\mathrm{U}(N)} \otimes V_R^{S_n} \tag{3.7}$$

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$S_m \times S_n$ is contained in centraliser of $\mathrm{U}(N)$ ($GL(N)$) acting in $V^{\otimes m} \otimes \bar{V}^{\otimes n}$. But we also need contractions, which along with the permutations, generate the algebra $B_N(m,n)$. Hence Schur-Weyl duality states that

$$V^{\otimes m} \otimes \bar{V}^{\otimes n} = \oplus_\gamma V_\gamma^{\mathrm{U}(N)} \otimes V_\gamma^{B_N(m,n)} \tag{3.8}$$

It gives the decomposition of the tensor product $V^{\otimes m} \otimes \bar{V}^{\otimes n}$ in terms of irreps of $\mathrm{U}(N)$ and $B_N(m,n)$. γ runs over sets of integers $(\gamma_1, \gamma_2, \cdots, \gamma_N)$ obeying $\gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_N$. The set of positive integers defines γ_+ which is a partition of $m-k$ while the negative integers define a partition γ_- of $n-k$. Here k is an integer lying between 0 and $min(m,n)$. Equivalently γ_+ determines a Young diagram with $m-k$ boxes, γ_- one of $n-k$ boxes. A choice of γ is equivalent to a choice of (k, γ_+, γ_-). If we write γ_+ as a Young diagram, with row lengths equal to the parts in the partition, $c_1(\gamma_+)$ is defined as the length of the first column. It follows from the above definitions that $c_1(\gamma_+) + c_1(\gamma_-) \leq N$. See more details on this in section 8.

From the definition of the Brauer algebra elements in terms of operators in tensor space, we can derive diagrammatic rules for multiplying them. The multiplication is done by stacking the diagrams corresponding to the Brauer elements. Symmetric group elements in S_n can be represented diagrammatically using two horizontal lines each containing n marked points labelled by integers $1 \ldots n$. We will refer to these as two rungs. Any particular element $\sigma \in S_n$ is represented by drawing lines joining an integer i from the bottom rung to an integer $\sigma(i)$ in the top rung. Multiplication of elements in S_n is obtained by stacking one pair of rungs on top of another, and identifying the top of the bottom pair to the bottom of the top pair. Elements in $S_m \times S_n$ are represented using two rungs with integers $1 \ldots m$ on the left side of a vertical barrier and $\bar{1}..\bar{n}$ on the right side of the vertical barrier. Brauer elements in $B_N(m,n)$ are drawn using two horizontal rungs as for $S_m \times S_n$, but now in addition to the lines of $S_m \times S_n$ we allow lines joining the points on the lower (upper) left of the barrier to points on the lower (upper) right of the barrier. Multiplication is done as before by stacking two pairs of rungs. Closed loops are replaced by the parameter N. This is illustrated in figures 4, 5. In figure 4 we have

$$(C_{3\bar{1}}(23)) \cdot (C_{3\bar{1}}(12)) = C_{3\bar{1}}(12)$$

In figure 5 we have

$$(C_{3\bar{1}}) \cdot (C_{3\bar{1}}(12)) = NC_{3\bar{1}}(12)$$

The Brauer algebra can be described by generators and relations. The relations can be obtained by diagrammatic manipulation. The generators include the simple transpositions s_i of S_m and $\bar{s}_i$ of S_n. The simple transposition s_i exchanges i with $i+1$ leaving everything else fixed. To this we add $C_{1\bar{1}}$ which contracts the first V factor with the first $\bar{V}$ factor. By using the diagrammatic approach, one easily derives relations such as

$$\begin{aligned} C_{i\bar{j}} &= (i1)(\bar{1}\bar{j})C_{1\bar{1}}(i1)(\bar{1}\bar{j}) \\ C_{i\bar{j}}(ik)C_{i\bar{j}} &= C_{i\bar{j}} \\ C_{i\bar{j}}C_{i\bar{k}} &= C_{i\bar{j}}(\bar{j}\bar{k}) = (\bar{j}\bar{k})C_{i\bar{k}} \\ C_{i\bar{j}}C_{k\bar{j}} &= C_{i\bar{j}}(ik) = (ik)C_{k\bar{j}} \end{aligned} \tag{3.9}$$

Of course these can also be derived equivalently by writing out all the operators involved in terms of their matrix elements in $V^{\otimes m} \otimes \bar{V}^{\otimes n}$.

It is also easy to check that Brauer elements commute with the action of the Lie algebra of $GL(N)$ or $\mathrm{U}(N)$ in $V \otimes \bar{V}$. Let E_{ij} be the matrix with 1 in the (i, j) entry, and 0 everywhere else. We have

$$\begin{aligned}(E_{ij} \otimes 1 + 1 \otimes E_{ij})\ C\ v_k \otimes \bar{v}_l &= (E_{ij} \otimes 1 + 1 \otimes E_{ij})\ \delta_{kl}\ v_m \otimes \bar{v}_m \\ &= \delta_{kl}(\delta_{jm} v_i \otimes \bar{v}_m - \delta_{im} v_m \otimes \bar{v}_j) \\ &= 0 \end{aligned} \tag{3.10}$$

and

$$\begin{aligned}C(E_{ij} \otimes 1 + 1 \otimes E_{ij}) v_k \otimes \bar{v}_l &= C(\delta_{jk} v_i \otimes \bar{v}_l - \delta_{il} v_k \otimes \bar{v}_j) \\ &= \delta_{jk}\delta_{il} v_m \otimes \bar{v}_m - \delta_{il}\delta_{jk} v_m \otimes \bar{v}_m) \\ &= 0 \end{aligned} \tag{3.11}$$

We record here the formula for dimensions of Brauer representations,

$$d_\gamma^{(B)} = \frac{m!n!}{k!(m-k)!(n-k)!} d_{\gamma_+} d_{\gamma_-} \tag{3.12}$$

in terms of the dimensions d_{γ_+} of the S_{m-k} representation associated with the partition γ_+, and d_{γ_-} of the S_{n-k} representation associated with the partition γ_-. There is also a useful formula for the multiplicity of an irrep (α, β) of $\mathbb{C}[S_m \times S_n]$ subalgebra of $B_N(m, n)$ appearing in the irrep γ of the Brauer algebra

$$M^\gamma_{(\alpha,\beta)} = \sum_{\delta \vdash k} g(\delta, \gamma_+; \alpha) g(\delta, \gamma_-; \beta) \tag{3.13}$$

Here $\delta \vdash k$ expresses the fact that δ is a partition of k. $g(\delta, \gamma_+; \alpha)$ is the Littlewood-Richardson (LR) coefficient, determined by putting together Young diagrams according to certain rules (see for example [34]). We will often use a single label A for the irreps of $\mathbb{C}[S_m \times S_n]$ where it is understood that $A = (\alpha \vdash m, \beta \vdash n)$, so that the multiplicity is written as M^γ_A.

3.3 The bilinear form and map between $B_N(m, n)$ to $\mathbb{C}[S_{m+n}]$

There is a symmetric bilinear form on the Brauer algebra, which follows by viewing them as operators in tensor space

$$< b_1, b_2 >= tr_{m,n}(b_1 b_2) \tag{3.14}$$

Above and in the rest of the paper $tr_{m,n}$ denotes the trace taken in $V^{\otimes m} \otimes \bar{V}^{\otimes n}$. Using the bilinear form we can define the dual element b^* of any element b by the property

$$tr_{m,n}(bb^*) = 1 \tag{3.15}$$

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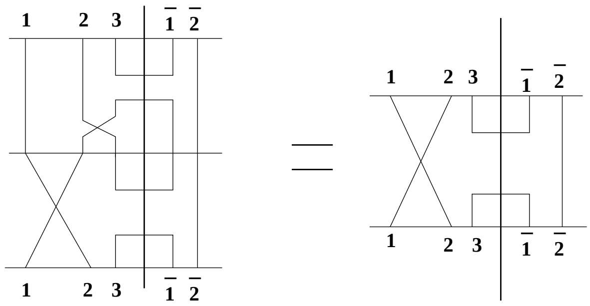

Figure 4: Example of product in Brauer algebra

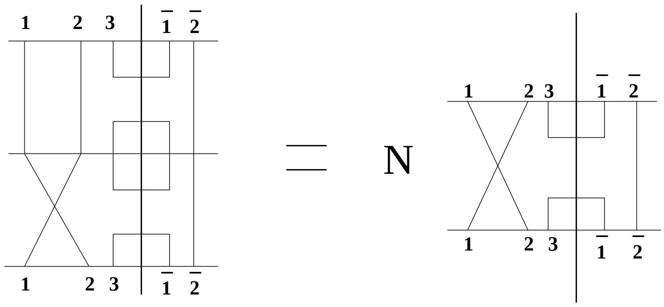

Figure 5: Example of product in Brauer algebra with loop giving N

It is shown in [30] that for any fixed element c the following sum

$$[c] = \sum_b bcb^* \tag{3.16}$$

over a complete basis of $B_N(m,n)$, gives a central element, which commutes with any $b \in B_N(m,n)$. This is a generalisation to semi-simple algebras of the group averaging procedure for group algebras. The dual elements also allow a construction of projectors

$$P^\gamma = t_\gamma \sum_b \chi_\gamma(b) b^* = t_\gamma \sum_b \chi_\gamma(b^*) b \tag{3.17}$$

where b runs over a basis for $B_N(m,n)$. The normalisation factor can be seen, in this case, to be $t_\gamma = Dim\gamma$, the dimension of the $\mathrm{U}(N)$ irrep associated with the label γ. This follows from [30] where it is shown P^γ is a central projector (idempotent), with t_γ equal to the trace of a matrix unit $tr(E^\lambda_{11})$. In our case the trace is being taken in $V^{\otimes m} \otimes \bar{V}^{\otimes n}$ and using Schur-Weyl duality (3.8) we find that

$$\begin{aligned} t_\gamma &= \sum_\lambda (Dim\lambda)\ tr_\lambda(E^\gamma_{11}) \\ &= \sum_\lambda (Dim\lambda)\ \delta_{\lambda\gamma} \\ &= Dim\gamma \end{aligned} \tag{3.18}$$

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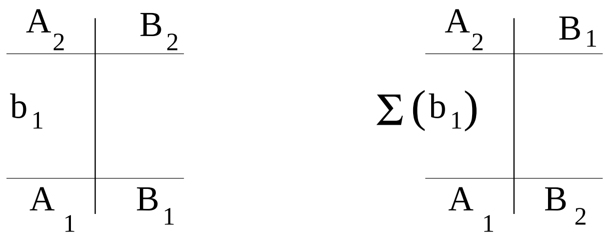

Figure 6: The map Σ from $B_N(m,n)$ to $\mathbb{C}[S_{m+n}]$

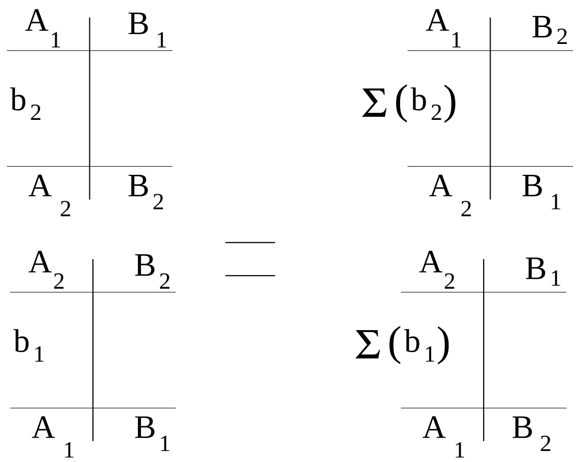

Figure 7: Showing that $tr(b_1 b_2) = tr(\Sigma(b_1)\Sigma(b_2))$

In the case at hand, we can obtain a lot of information about the bilinear form (3.14) by exploiting a map Σ

$$\Sigma : B_N(m,n) \to \mathbb{C}[S_{m+n}] \tag{3.19}$$

We recall that in the diagrammatic description of Brauer elements given earlier in this section, we have two horizontal lines, one on top of the other. Each line has $m+n$ points, with a vertical barrier separating the m from the n. In $B_N(m,n)$ lines crossing the barrier join points on the upper left to points on the upper right, and points on the lower left points on the lower right. Elements on S_{m+n} are described by lines joining lower points to upper points whether they cross the barrier or not. The map Σ simply reflects the upper right segment and the lower right segment into each other. It is illustrated in figure 6, where B_1, B_2 denote the sets of points labelled $1..n$.

It is an invertible map from $B_N(m,n)$ to $\mathbb{C}[S_{m+n}]$, which is consistent with the fact that the dimension of the Brauer algebra as a vector space is known to be $(m+n)!$ (e.g see [27]). It is not a homomorphism. It however has the crucial property that it maps the symmetric bilinear form on $B_N(m,n)$ to a symmetric bilinear form on $\mathbb{C}[S_{m+n}]$

$$tr_{m,n}(b_i b_j) = tr_{m,n}(\Sigma(b_i)\Sigma(b_j)) \tag{3.20}$$

where b_i runs over a complete basis for the Brauer algebra. This is made clear by figure 7. On the left hand side, the set of points labelled by B_2 (a set of labels for tensor space indices) on the upper diagram for b_2 is identified with the set B_2 on the lower diagram for b_1 by the multiplication $b_2 b_1$. The sets labelled by B_1 are identified by the trace. On the right hand side, the Σ map performs the reflection of labels $B_1 \leftrightarrow B_2$. The multiplication of $\Sigma(b_2)$ with $\Sigma(b_1)$ identifies the B_1 sets. The trace identifies the B_2 sets. The outcome in both cases is determined by the B_1, B_2 identifications, which proves (3.20). An explicit formula for the bilinear form follows

$$\begin{aligned} tr(\Sigma(b_i)\Sigma(b_j)) &= \sum_{T \vdash m+n} DimT \; \chi_T(\Sigma(b_i)\Sigma(b_j)) \\ &= N^{m+n}\delta(\Omega_{m+n}\Sigma(b_i)\Sigma(b_j)) \end{aligned} \tag{3.21}$$

The delta function over the symmetric group algebra $\delta(\sigma)$ is defined to be 1 if σ is the identity and 0 otherwise. For $m+n \geq N$, the sum over T is restricted by the condition $c_1(T) \leq N$. The Ω_{m+n} factor is familiar from 2dYM theory, and is defined by

$$\Omega_n = \sum_{\sigma \in S_n} N^{C_\sigma - n} \sigma \tag{3.22}$$

where C_σ is the number of cycles in the permutation σ. When $n < N$, Ω_n can be inverted, and this is used to good effect in the large N expansion of 2dYM. Here we have

$$(\Sigma(b))^* = N^{-m-n}\Omega^{-1}_{m+n}(\Sigma(b))^{-1} \tag{3.23}$$

Since Σ preserves the bilinear form (3.20) we have $(\Sigma(b))^* = \Sigma(b^*)$

$$b^* = N^{-m-n}\Sigma^{-1}(\Omega^{-1}_{m+n}(\Sigma(b))^{-1}) \tag{3.24}$$

An expansion of the Ω^{-1}_{m+n} in terms of characters can be given

$$\begin{aligned} \Omega^{-1}_{m+n} &= \frac{N^{m+n}}{(m+n)!}\sum_T \frac{d_T}{DimT} p_T \\ &= \frac{N^{m+n}}{((m+n)!)^2}\sum_T \frac{d_T^2}{DimT}\sum_{\sigma \in S_{m+n}} \chi_T(\sigma)\;\sigma \end{aligned} \tag{3.25}$$

so that

$$(\Sigma(b))^* = \frac{1}{((m+n)!)^2}\sum_T \frac{d_T^2}{DimT}\sum_{\sigma \in S_{m+n}} \chi_T(\sigma)\sigma(\Sigma(b))^{-1} \tag{3.26}$$

and

$$b^* = \frac{1}{((m+n)!)^2}\sum_T \frac{d_T^2}{DimT}\sum_{\sigma \in S_{m+n}} \chi_T(\sigma)\;\Sigma^{-1}(\;\sigma(\Sigma(b))^{-1}\;) \tag{3.27}$$

For $m+n \leq N$ the sum T runs over all partitions of $m+n$, but more generally $c_1(T) \leq N$.

It is instructive to consider

$$G_{ij} = tr_{m,n}(b_i b_j) = tr_{m,n}(\sigma_i \sigma_j) \tag{3.28}$$

where b_i belongs to a basis set in $B_N(m,n)$ while σ_i is the corresponding element $\Sigma(b_i)$ in S_{m+n}. The inverse is defined as

$$\sigma_i^* = G^{ij}\sigma_j \tag{3.29}$$

From this equation, we obtain

$$\delta(\sigma_i^* \sigma_k^{-1}) = G^{ij}\delta(\sigma_j\sigma_k^{-1}) = G^{ik} \tag{3.30}$$

$$\begin{aligned} G^{ij} &= \delta(\sigma_i^*\sigma_j^{-1}) \\ &= \frac{1}{N^{m+n}}\delta\left(\Omega_{m+n}^{-1}\sigma_i^{-1}\sigma_j^{-1}\right) \\ &= \frac{1}{((m+n)!)^2}\sum_T \frac{d_T^2}{dimT}\chi_T(\sigma_i^{-1}\sigma_j^{-1}) \end{aligned} \tag{3.31}$$

As usual at finite N we restrict $c_1(T) \leq N$. We have used the following equations:

$$\begin{aligned} \delta(\sigma) &= \frac{1}{(m+n)!}\sum_T d_T\chi_T(\sigma) \\ dimT &= \frac{N^{m+n}}{(m+n)!}\chi_T(\Omega_{m+n}) \end{aligned} \tag{3.32}$$

3.4 Brauer algebra as a spectrum generating algebra for multi-traces of two matrices

In the case of holomorphic multi-trace operators constructed from n copies of Φ it was useful to write them as $tr_n(\sigma\Phi)$, where the trace is in $V^{\otimes n}$ [1]. Different multi-trace operators correspond to different states in the Hilbert space of $N = 4$ SYM on $S^3 \times R$, via the operator-state correspondence (for elaboration of this see [7, 35, 36]). When two permutations σ_1, σ_2 in S_n are related by a permutation σ_3 as $\sigma_1 = \sigma_3\sigma_2\sigma_3^{-1}$, they lead to the same multi-trace operator, and the same state in the Hilbert space in $N = 4$ SYM. Permutations $\sigma \in S_n$, subject to an equivalence relation of conjugation by another element in S_n, are just the conjugacy classes of S_n. In each equivalence class we have a central element: the sum of permutations in the conjugacy class which is proportional to the group average $\sum_{\sigma_1}\sigma_1\sigma\sigma_1^{-1}$. The conjugacy classes of S_n can be viewed, therefore, as forming a spectrum generating algebra.

In the non-chiral case at hand, where we are considering multi-traces constructed from $\Phi, \Phi^\dagger$, the Brauer algebra plays an analogous role. In the simplest case of $B_N(1,1)$, the Brauer algebra is two-dimensional, spanned by 1 and C. We have

$$tr_{(1,1)}(\Phi \otimes \Phi^*) = tr\Phi tr\Phi^\dagger \tag{3.33}$$

and

$$\begin{aligned} tr_{(1,1)}C(\Phi\otimes\Phi^*) &= < e^i \otimes \bar{e}^j|C(\Phi\otimes\Phi^*)|e_i\otimes\bar{e}_j > \\ &= < e^i \otimes \bar{e}^j|C\Phi_i^k\Phi_j^{*l}|e_k\otimes e_l > \\ &= < e^i \otimes \bar{e}^j|\delta_{kl}\Phi_i^k\Phi_j^{*l}|e_m\otimes\bar{e}_m > \\ &= tr\Phi\Phi^\dagger \end{aligned} \tag{3.34}$$

In fact any multi-trace constructed from $\Phi, \Phi^\dagger$ can be obtained by using $tr(b\ \Phi \otimes \Phi^\dagger)$ for general b. Consider any trace $tr_{m,n}(\Phi^{m_1}\Phi^{\dagger\ n_1}\Phi^{m_2}\Phi^{\dagger\ n_2}\dots\Phi^{m_k}\Phi^{\dagger\ n_k})$. The element b in this case involves cyclic permutations with cycle lengths $m_1, m_2\dots$ in S_m and a permutation with cycles $n_1, n_2\dots$ in S_n along with k contractions and a further permutation to join the contractions. One can write an explicit formula to demonstrate the above, but the reader should easily convince him(her)self of the above claim by considering a few examples. An important point to note is that a Brauer element b and another element hbh^{-1} produce the same multi-trace if $h \in S_m \times S_n$. In the diagrammatic presentation of Brauer elements, the conjugation corresponds to re-labelling the numbers on the top and bottom rungs. Therefore, counting multi-traces is the same as counting these equivalence classes under conjugation by h. In each equivalence class we can build, by averaging any chosen element b using $\sum_h hbh^{-1}$, a unique element which commutes with $S_m \times S_n$. We will call such elements, *symmetric elements*. It is interesting that this notion of equivalence by conjugation with $h \in S_m \times S_n$ has been studied in [27] purely as an algebraic property. Here the equivalence is motivated by the role of Brauer as a spectrum generating algebra for multi-traces.

A natural way to construct symmetric elements from representation theory is to consider projectors for fixed irreducible representations. In sections 4, 5 we will be considering projectors for Brauer irreps. Since each Brauer irrep decomposes under the action of the $\mathbb{C}[S_m \times S_n]$ sub-algebra into irreducible reps according to (3.13), the central Brauer projectors will be sums of *symmetric Brauer projectors*. However the symmetric Brauer projectors do not exhaust the complete set of symmetric elements. One has to consider more general *symmetric branching operators*. We will describe them in more detail in section 7, where we will also argue that they provide a complete set of symmetric elements. In section 6, we will show how to count symmetric projectors, and symmetric branching operators, and show agreement with the known counting of large N multi-traces based on Polya theory. The Brauer counting however extends beyond the large N limit, and finite N effects will be described in section 8.

We observe some useful facts, which involve the map Σ given in section 3.3. The map Σ is also useful when we construct gauge invariant operators from the matrices $\Phi, \Phi^\dagger$. The equation

$$tr_{m,n}(b\Phi \otimes \Phi^*) = tr_{m,n}(\Sigma(b)\Phi \otimes \Phi^\dagger) \tag{3.35}$$

follows from the diagrammatics. In the Matrix quantum mechanics we have the related fact that

$$tr_{m,n}(\Sigma(b)A^\dagger \otimes B^\dagger) = tr_{m,n}(bA^\dagger \otimes (B^\dagger)^T) \tag{3.36}$$

The transpose acts on the matrix indices. It is also useful to note that

$$hbh^{-1} = b \quad \Longleftrightarrow \quad h\Sigma(b)h^{-1} = \Sigma(b) \tag{3.37}$$

In other words, if b is a symmetric element, then $\Sigma(b)$ is a symmetric element.

4. Projectors in Brauer algebra

As reviewed in the discussion of (3.8) the tensor product $V^{\otimes m} \otimes \bar{V}^{\otimes n}$ decomposes into a direct sum of irreps of $B_N(m,n)$ and $\mathrm{U}(N)$ labelled by γ, where $\gamma = (k, \gamma_+, \gamma_-)$. For each γ, there is a projector P^γ which, acting on $V^{\otimes m} \otimes \bar{V}^{\otimes n}$, projects onto the subspace labelled by γ. This projector can be constructed from Brauer elements, and is central, i.e commutes with any element of $B_N(m,n)$. The $k=0$ projectors have special properties. Writing $P^{(k=0,\gamma_+=R,\gamma_-=S)} \equiv P_{R\bar{S}}$ to connect to the notation of 2D Yang Mills theory, we have, for a unitary matrix U,

$$tr_{m,n}(P_{R\bar{S}}U) = d^{(B)}_{R\bar{S}}\chi_{R\bar{S}}(U) = d_R d_S \chi_{R\bar{S}}(U) \tag{4.1}$$

On $V^{\otimes m} \otimes \bar{V}^{\otimes n}$, U acts as $U \otimes U \cdots U \otimes U^* \otimes \cdots U^*$ with m factors of U and n factors of U^*. The first equality follows from (3.8) and the second equality expresses the Brauer dimension using (3.12) in terms of the symmetric group irrep dimensions d_R, d_S. The character $\chi_{R\bar{S}}(U)$ is the character of the "coupled representation" used in 2dYM [21, 22]. By setting $U = 1$ we have

$$tr_{m,n}(P_{R\bar{S}}) = d_R d_S Dim R\bar{S} \tag{4.2}$$

In section 7 we will be considering local operators in 4D field theory $tr_{m,n}(P_{R\bar{S}}\Phi \otimes \Phi^*)$ as well as related operators in the reduced Matrix quantum mechanics. The notation P^γ will be used for general γ, $P_{R\bar{S}}$ being reserved for the special case of γ with $k=0$. Note that, since Φ is a general complex matrix, not necessarily satisfying $\Phi\Phi^\dagger = 1$, $tr_{m,n}(P_{R\bar{S}}\Phi \otimes \Phi^*)$ cannot be obtained just by replacing $U \to \Phi$ in $\chi_{R\bar{S}}(U)$.

In this section and the next, we will be describing various formulae for the construction of the orthogonal set of central projectors P^γ. For $k \neq 0$, the P^γ will be a sum of orthogonal *symmetric projectors*.

$$P^\gamma = \sum_{A,i} P^\gamma_{A,i} \tag{4.3}$$

The symmetric projectors $P^\gamma_{A,i}$ are not, in general, Brauer central elements, but they do commute with elements h in the $\mathbb{C}[S_m \times S_n]$ subalgebra of $B_N(m,n)$. A labels irreps of $\mathbb{C}[S_m \times S_n]$.

4.1 Projector for $k = 0$ using character formula

Starting from (3.17) we rewrite the projector for $k=0$ using the character formula for elements of Brauer algebra, which is in theorem 7.20 in [27]

$$\chi^\gamma_{B_N(m,n)}(\zeta) = N^h \sum_{\lambda \vdash m', \pi \vdash n'} \left(\sum_{\delta \vdash (k-h)} g(\delta, \gamma^+; \lambda) g(\delta, \gamma^-; \pi) \right) \chi^\lambda_{S_{m'}}(\zeta^+) \chi^\pi_{S_{n'}}(\zeta^-) \tag{4.4}$$

where $\gamma^+ \vdash (m-k)$, $\gamma^- \vdash (n-k)$, $\zeta^+ \vdash m' = (m-h)$ and $\zeta^- \vdash n' = (n-h)$. ζ denotes a element of Brauer algebra, and we use b hereafter instead of ζ. h is an integer which is determined by the minimal number of contractions in a Brauer element. For $b = \sigma \otimes \tau \in \mathbb{C}[S_m \times S_n]$ we have $h = 0$ and $h \geq 1$ if b contains contractions.

We set $k = 0$ in the character formula (4.4). In this case, $\gamma^+ \equiv R \vdash m$, $\gamma^- \equiv S \vdash n$. For b with $h \geq 1$, $k - h = -h < 0$, so it cannot have any partitions δ, hence relevant characters vanish. Therefore $\chi_\gamma(b) \neq 0$ only for $b \in S_m \times S_n$ when $k = 0$. Then the character of $b = \sigma \otimes \tau \in S_m \times S_n$ can be calculated as

$$\begin{aligned}
\chi^{\gamma}_{B_N(m,n)}(\sigma \otimes \tau) &= \sum_{\lambda \vdash m, \pi \vdash n} g(\emptyset, R; \lambda) g(\emptyset, S; \pi) \chi^{\lambda}_{S_m}(\sigma) \chi^{\pi}_{S_n}(\tau) \\
&= \chi^{R}_{S_m}(\sigma) \chi^{S}_{S_n}(\tau)
\end{aligned} \tag{4.5}$$

We also use

$$b^* = (1^*) b^{-1} \quad b \in \mathbb{C}[S_m \times S_n] \tag{4.6}$$

which is a special case of (3.24). We now rewrite the projector for $k = 0$ using the above things.

$$\begin{aligned}
P_{R\bar{S}} &= DimR\bar{S} \sum_b \chi_\gamma(b) b^* \\
&= DimR\bar{S} \sum_{b \in S_m \times S_n} \chi_\gamma(b) b^* \\
&= DimR\bar{S} \sum_{\sigma \in S_m, \tau \in S_n} \chi_R(\sigma) \chi_S(\tau) 1^* (\sigma \otimes \tau)^{-1} \\
&= DimR\bar{S} 1^* \sum_{\sigma \in S_m} \chi_R(\sigma) \sigma^{-1} \sum_{\tau \in S_n} \chi_S(\tau) \tau^{-1} \\
&= DimR\bar{S} \frac{m!n!}{d_R d_S} 1^* p_R \bar{p}_S
\end{aligned} \tag{4.7}$$

Using

$$\begin{aligned}
1^* &= \frac{1}{N^{m+n}} \Sigma^{-1} \left(\Omega^{-1}_{m+n} \right) \\
&= \frac{1}{(m+n)!} \sum_T \frac{d_T}{dimT} \Sigma^{-1} (p_T) \\
&= \frac{1}{((m+n)!)^2} \sum_{T \vdash m+n} \frac{d_T^2}{dimT} \sum_\sigma \chi_T(\sigma^{-1}) \Sigma^{-1} (\sigma)
\end{aligned} \tag{4.8}$$

where we have used (3.25) to obtain the second line, we have

$$P_{R\bar{S}} = \frac{m!n!}{((m+n)!)^2} \frac{DimR\bar{S}}{d_R d_S} \sum_{T \vdash m+n} \frac{d_T^2}{dimT} \sum_\sigma \chi_T(\sigma^{-1}) \Sigma^{-1} (\sigma)\, p_R \bar{p}_S \tag{4.9}$$

In appendix A.1, we will use the expression (4.9) to obtain some examples of projectors. If we take a trace of the expression for $P_{R\bar{S}}$ we obtain

$$\frac{(m+n)!}{m!n!} d_R d_S = \sum_T g(R, S; T) d_T \tag{4.10}$$

which we know to be a true identity from facts about induced representations from $S_m \times S_n$ to S_{m+n} [34]. This gives a check of the validity of $t_\gamma = DimR\bar{S}$.

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4.2 Relation between $B_N(m,n)$ and $\mathbb{C}[S_{m+n}]$ and a new formula for dimension of coupled representations

Starting with the form

$$P_\gamma = t_\gamma \sum_b \chi_\gamma(b^*)b$$

we can write the projector as

$$P_\gamma = (Dim\gamma)\sum_\sigma \Sigma^{-1}(\sigma)N^{-m-n}\chi_\gamma(\Sigma^{-1}(\Omega_{m+n}^{-1}\cdot\sigma^{-1})) \tag{4.11}$$

For the $k=0$ representations,

$$P_{R\bar{S}} = (DimR\bar{S})\sum_\sigma N^{-m-n}\chi_{R\otimes S}(\Sigma^{-1}(\Omega_{m+n}^{-1}\cdot\sigma^{-1})|_{S_m\times S_n}) \;\; \Sigma^{-1}(\sigma) \tag{4.12}$$

We know that the term without contractions is $p_R\otimes p_S$. The coefficient of 1 is $\frac{d_R^2 d_S^2}{m!n!}$. By equating this to the term obtained from (4.12) by setting $\sigma = 1$, we have

$$\frac{d_R^2 d_S^2}{m!n!} = N^{-m-n}\; DimR\bar{S}\; \chi_{R\otimes S}(\Sigma^{-1}(\Omega_{m+n}^{-1})|_{S_m\times S_n}) \tag{4.13}$$

Using (3.25) and restricting to the subgroup

$$\Omega_{m+n}^{-1}|_{S_m\times S_n} = \frac{N^{m+n}}{((m+n)!)^2}\sum_T \frac{d_T^2}{DimT}\sum_{\sigma_1\in S_m}\sum_{\sigma_2\in S_n}\chi_T(\sigma_1\cdot\sigma_2)\sigma_1\cdot\sigma_2 \tag{4.14}$$

The character $\chi_T(\sigma_1\cdot\sigma_2)$ can be expanded using LR coefficients:

$$\chi_T(\sigma_1\cdot\sigma_2) = \sum_{R_1,S_1} g(R_1,S_1;T)\chi_{R_1}(\sigma_1)\chi_{S_1}(\sigma_2) \tag{4.15}$$

We also need to use

$$\begin{aligned}\Sigma^{-1}(\sigma_1\cdot\sigma_2) &= \sigma_1\cdot\sigma_2^{-1}\\ \chi_{R\otimes S}(\sigma_1\cdot\sigma_2^{-1}) &= \chi_R(\sigma_1)\chi_S(\sigma_2)\end{aligned} \tag{4.16}$$

and the orthogonality of characters

$$\frac{1}{m!}\sum_{\sigma_1}\chi_R(\sigma_1)\chi_{R_1}(\sigma_1) = \delta_{RR_1} \tag{4.17}$$

Using these facts to simplify the r.h.s. of (4.13)

$$\frac{d_R^2 d_S^2}{DimR\bar{S}} = \frac{m!^2 n!^2}{(m+n)!^2}\sum_T \frac{d_T^2}{DimT}g(R,S;T) \tag{4.18}$$

For $m+n\le N$, T above runs over all Young diagrams with $m+n$ boxes. In the case $m+n > N$, but with the condition $c_1(R)+c_1(S) < N$ which is necessary for $P_{R\bar{S}}$ to exist, the formula (4.18) is still valid but T now runs over all Young diagrams with no more than N rows, or equivalently $c_1(T)\le N$. We can check (4.18) easily in cases such as $(R,S) = ([1],[1]),([1],[2]),([1],[2,1]),([2],[2,1])$.

5. Examples of projectors

In this section, we give some examples of projectors. The derivation of $k = 0$ projectors for some cases based on (4.9) is given in appendix A.1.

5.1 $V^{\otimes m} \otimes \bar{V}$

We will now give the general $k = 0$ central projector for $B_N(m, 1)$

$$P_{R[\bar{1}]} = \left(1 - \frac{1}{N\Omega_m} \sum_i \Omega_{m-1}^{<i>} C_{i\bar{1}}\right) p_R \tag{5.1}$$

$\Omega_{m-1}^{<i>}$ is the omega factor for the i-th embedding of S_{m-1} in S_m, where the i-th index is removed from S_m. $\Omega_{m-1}^{<i>}$ satisfies

$$\Omega_{m-1}^{<i>} C_{i\bar{1}} = C_{i\bar{1}} \Omega_{m-1}^{<i>} \tag{5.2}$$

and

$$(ki)\Omega_{m-1}^{<i>} = \Omega_{m-1}^{<k>}(ki) \tag{5.3}$$

If we use

$$\Omega_m = \Omega_{m-1}^{<k>} \left(1 + \frac{1}{N} \sum_{i \neq k}^{m} (ik)\right) \tag{5.4}$$

which was found useful in the study of loop equations in 2dYM [26], we get another expression of the projector

$$P_{R[\bar{1}]} = \left(1 - \frac{1}{N + \sum_{i \neq k}^{m} (ik)} C_{i\bar{1}}\right) p_R \tag{5.5}$$

We show that the projector satisfies $C_{k\bar{1}} P_{R[\bar{1}]} = 0$. First, we obtain the following equation

$$\begin{aligned}
C_{k\bar{1}} \sum_i \Omega_{m-1}^{<i>} C_{i\bar{1}} &= C_{k\bar{1}} \left(\Omega_{m-1}^{<k>} C_{k\bar{1}} + \sum_{i \neq k} \Omega_{m-1}^{<i>} C_{i\bar{1}}\right) \\
&= \Omega_{m-1}^{<k>} N C_{k\bar{1}} + \sum_{i \neq k} C_{k\bar{1}} (ki) \Omega_{m-1}^{<i>} \\
&= C_{k\bar{1}} \Omega_{m-1}^{<k>} N \left(1 + \frac{1}{N} \sum_{i \neq k} (ki)\right) \\
&= C_{k\bar{1}} \Omega_m N
\end{aligned} \tag{5.6}$$

where we have used (5.4). Using this equation, it is easy to show

$$\begin{aligned}
C_{k\bar{1}} P_{R[\bar{1}]} &= \left(C_{k\bar{1}} - \frac{1}{N\Omega_m} C_{k\bar{1}} \sum_i \Omega_{m-1}^{<i>} C_{i\bar{1}}\right) p_R \\
&= \left(C_{k\bar{1}} - \frac{1}{N\Omega_m} C_{k\bar{1}} \Omega_m N\right) p_R \\
&= 0
\end{aligned} \tag{5.7}$$

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We will now show, using (4.2), that this agrees with the Gross-Taylor dimension formula [22]. The trace of the first term in (5.1) is

$$\begin{aligned} tr_{m,1}(p_R) &= \frac{d_R}{m!}\sum_R \chi_R(\sigma) tr_{m,1}(\sigma) \\ &= \frac{d_R}{m!}\sum_R \chi_R(\sigma) N^{K_\sigma+1} \end{aligned} \tag{5.8}$$

The trace of the second term in (5.1) is

$$\begin{aligned} tr_{m,1}\left(\frac{1}{N\Omega_m}\Omega_{m-1}^{<i>}C_{i\bar{1}}p_R\right) &= \frac{d_R}{N\chi_R(\Omega_m)}\sum_{i,j} tr_{m,1}(\Omega_{m-1}^{<i>}C_{i\bar{1}}p_R) \\ &= \frac{d_R}{N\chi_R(\Omega_m)}\sum_i tr_m(\Omega_{m-1}^{<i>}p_R) \\ &= \frac{d_R}{N\chi_R(\Omega_m)} DimR \sum_i \chi_R(\Omega_{m-1}^{<i>}) \\ &= \frac{d_R}{m!}N^{m-1}\sum_i \chi_R(\Omega_{m-1}^{<i>}) \\ &= \frac{d_R}{m!}\sum_\sigma \chi_R(\sigma) N^{K_\sigma+1}\sigma_1 \frac{1}{N^2} \end{aligned} \tag{5.9}$$

In the last step, we have used the following equation,

$$\begin{aligned} \sum_{i=1}^m \chi_R(\Omega_{m-1}^{<i>}) &= \sum_{i=1}^m \sum_{\sigma\in S_{m-1}^{<i>}} \chi(\sigma) N^{K_\sigma - m} \\ &= \sum_{\sigma\in S_m} \sigma_1 \chi(\sigma) N^{K_\sigma - m}. \end{aligned} \tag{5.10}$$

where σ_1 is the number of 1-cycles in σ. Hence

$$tr_{m,1}(P_{R[\bar{1}]}) = \frac{d_R}{m!}\sum_R \chi_R(\sigma) N^{K_\sigma+1}\left(1 - \frac{\sigma_1}{N^2}\right) \tag{5.11}$$

which can be recognised as $d_R DimR[\bar{1}]$ using [22].

5.2 Specific examples: $V^{\otimes 2}\otimes \bar{V}$

In this case, we have two $k=0$ projectors

$$\begin{aligned} P_{[2][\bar{1}]} &= \left(1 - \frac{1}{N+1}C\right)p_{[2]} \\ P_{[1^2][\bar{1}]} &= \left(1 - \frac{1}{N-1}C\right)p_{[1^2]} \end{aligned} \tag{5.12}$$

where $C \equiv C_{1\bar{1}} + C_{2\bar{1}}$ commutes with any element in $\mathbb{C}[S_2]$. A $k=1$ projector is given by the sum of the second terms of $k=0$ projectors

$$P^{(k=1,\gamma_+=[1],\gamma_-=\emptyset)} = \frac{1}{N+s}C = \sum_R P_{R[\bar{1}]}^{(k=1,\gamma_+=[1],\gamma_-=\emptyset)} \tag{5.13}$$

where

$$
\begin{aligned}
P^{(k=1,\gamma_+=[1],\gamma_-=\emptyset)}_{[2][\bar{1}]} &= \frac{1}{N+1} C p_{[2]} \\
P^{(k=1,\gamma_+=[1],\gamma_-=\emptyset)}_{[1^2][\bar{1}]} &= \frac{1}{N-1} C p_{[1^2]}
\end{aligned} \tag{5.14}
$$

which are symmetric projectors.

5.3 Specific examples: $V^{\otimes 3} \otimes \bar{V}$

In this case, we have three $k = 0$ projectors

$$
\begin{aligned}
P_{[3][\bar{1}]} &= \left(1 - \frac{1}{N+2} C\right) p_{[3]} \\
P_{[1^3][\bar{1}]} &= \left(1 - \frac{1}{N-2} C\right) p_{[1^3]} \\
P_{[2,1][\bar{1}]} &= \left(1 - \frac{N}{(N^2-1)} C - \frac{1}{(N^2-1)} D\right) p_{[2,1]}
\end{aligned} \tag{5.15}
$$

where $C \equiv C_{1\bar{1}} + C_{2\bar{1}} + C_{3\bar{1}}$ and $D \equiv C_{1\bar{1}} s_2 + C_{2\bar{1}} s_1 s_2 s_1 + C_{3\bar{1}} s_1$ commute with any element in $\mathbb{C}[S_3]$. Some useful formulae are given in appendix A.2.1.

We have two $k = 1$ projectors

$$
\begin{aligned}
P^{(k=1,\gamma_+=[2],\gamma_-=\emptyset)} &= \frac{1}{N+2} C p_{[3]} + \frac{1}{2} \frac{1}{N-1} (C + D) p_{[2,1]} \\
&= P^{(k=1,\gamma_+=[2],\gamma_-=\emptyset)}_{[3][\bar{1}]} + P^{(k=1,\gamma_+=[2],\gamma_-=\emptyset)}_{[2,1][\bar{1}]} \\
P^{(k=1,\gamma_+=[1^2],\gamma_-=\emptyset)} &= \frac{1}{N-2} C p_{[1^3]} + \frac{1}{2} \frac{1}{N+1} (C - D) p_{[2,1]} \\
&= P^{(k=1,\gamma_+=[1^2],\gamma_-=\emptyset)}_{[1^3][\bar{1}]} + P^{(k=1,\gamma_+=[1^2],\gamma_-=\emptyset)}_{[2,1][\bar{1}]}
\end{aligned} \tag{5.16}
$$

Each term in the r.h.s. is a symmetric projector. Therefore, we have seven symmetric operators and five central Brauer projectors. The decomposition of central Brauer projectors into symmetric projectors in (5.16) can be understood in terms of the decomposition of Brauer irreps in terms $\mathbb{C}[S_3 \times S_1]$ irreps. For example, we know using (3.13) that the $(k = 1, \gamma_+ = [2], \gamma_- = \emptyset)$ irrep of $B_N(3, 1)$ contains the direct sum of irreps $([3], [1])$ and $([2, 1], [1])$ of $\mathbb{C}[S_3 \times S_1]$, each with unit multiplicity.

5.4 Specific examples: $V^{\otimes 2} \otimes \bar{V}^{\otimes 2}$

We have four $k = 0$ projectors

$$
P_{R\bar{S}} = \left(1 - \frac{1}{(N+s+\bar{s})} C_{(1)} + \frac{1}{(N+s)(N+s+\bar{s})} C_{(2)}\right) p_R \bar{p}_S \tag{5.17}
$$

where $R = [2]$ or $[1^2]$, $S = [2]$ or $[1^2]$, $s = (12)$, $\bar{s} = (\bar{1}\bar{2})$ and

$$
C_{(1)} = C_{1\bar{1}} + C_{1\bar{2}} + C_{2\bar{1}} + C_{2\bar{2}} \quad C_{(2)} = C_{1\bar{1}} C_{2\bar{2}} + C_{1\bar{2}} C_{2\bar{1}} \tag{5.18}
$$

which commute with any element in $\mathbb{C}[S_2 \times S_2]$. Some useful formulae are given in appendix A.2.2. $k \neq 0$ central Brauer projectors are given by

$$\begin{aligned} P^{(k=1,\gamma_+=[1],\gamma_-=[1])} &= \frac{1}{(N+s+\bar{s})}\left(C_{(1)} - \frac{2}{N}C_{(2)}\right) \\ P^{(k=2,\gamma_+=\emptyset,\gamma_-=\emptyset)} &= \frac{1}{N(N+s)}C_{(2)} \end{aligned} \tag{5.19}$$

These two $k \neq 0$ central Brauer projectors can be written as the sum of symmetric projectors as

$$\begin{aligned} P^{(k=1,\gamma_+=[1],\gamma_-=[1])} &= \sum_{R,S} P^{(k=1,\gamma_+=[1],\gamma_-=[1])}_{R\bar{S}} \\ P^{(k=2,\gamma_+=\emptyset,\gamma_-=\emptyset)} &= \sum_{R,S} P^{(k=2,\gamma_+=\emptyset,\gamma_-=\emptyset)}_{R\bar{S}} \end{aligned} \tag{5.20}$$

where

$$\begin{aligned} P^{(k=1,\gamma_+=[1],\gamma_-=[1])}_{R\bar{S}} &= \frac{1}{(N+s+\bar{s})}\left(C_{(1)} - \frac{2}{N}C_{(2)}\right)p_R\bar{p}_S \\ P^{(k=2,\gamma_+=\emptyset,\gamma_-=\emptyset)}_{R\bar{S}} &= \frac{1}{N(N+s)}C_{(2)}p_R\bar{p}_S \end{aligned} \tag{5.21}$$

Because of

$$C_{(2)}p_{[2]}\bar{p}_{[1,1]} = C_{(2)}p_{[1,1]}\bar{p}_{[2]} = 0 \tag{5.22}$$

which can be easily checked using $C_{(2)}s = C_{(2)}\bar{s}$, $P^{(k=2,\gamma_+=\emptyset,\gamma_-=\emptyset)}_{[2][\bar{1^2}]}$ and $P^{(k=2,\gamma_+=\emptyset,\gamma_-=\emptyset)}_{[1^2][\bar{2}]}$ vanish. Therefore we have ten symmetric projectors and six central Brauer projectors.

5.5 Specific examples: composites of symmetric and anti-symmetric

We write down projectors corresponding to an AdS giant and an AdS anti-giant $([m],[n])$, an S-giant and an S-anti-giant $([1^m],[1^n])$, and a composite of an AdS giant and an S anti-giant $([m],[1^n])$ or vice versa $([1^m],[n])$. We assume $m \geq n$ in this subsection.

We first define

$$\begin{aligned} C_{(1)} &= \sum_{ij} C_{i\bar{j}} \quad C_{(2)} = \frac{1}{2!}\sum_{i\neq j}\sum_{k\neq l} C_{i\bar{k}}C_{j\bar{l}} \quad \cdots \\ C_{(k)} &= \frac{1}{k!}\sum_{i_a\neq i_b}\sum_{j_a\neq j_b} C_{i_1\bar{j}_1}C_{i_2\bar{j}_2}\cdots C_{i_k\bar{j}_k} \end{aligned} \tag{5.23}$$

Using these, we obtain projectors for $k = 0$ representations:
$(R,S) = ([m],[n])$

$$\begin{aligned} P_{[m][\bar{n}]} &= \Bigg(1 - \frac{1}{N+m+n-2}C_{(1)} \\ &\qquad + \frac{1}{(N+m+n-3)(N+m+n-2)}C_{(2)} + \cdots \Bigg) p_{[m]}\bar{p}_{[n]} \\ &= \left(1 + \sum_{k=1}^{n}(-1)^k \prod_{l=1}^{k} \frac{1}{(N+m+n-l-1)}C_{(k)}\right) p_{[m]}\bar{p}_{[n]} \end{aligned} \tag{5.24}$$

$(R, S) = ([1^m], [1^n])$

$$P_{[1^m][\bar{1^n}]} = \left(1 + \sum_{k=1}^{n} (-1)^k \prod_{l=1}^{k} \frac{1}{(N - m - n + l + 1)} C_{(k)}\right) p_{[1^m]} \bar{p}_{[1^n]} \tag{5.25}$$

$(R, S) = ([m], [1^n])$

$$P_{[m][\bar{1^n}]} = \left(1 + \sum_{k=1}^{n} (-1)^k \prod_{l=1}^{k} \frac{1}{(N + m - n)} C_{(k)}\right) p_{[m]} \bar{p}_{[1^n]} \tag{5.26}$$

$(R, S) = ([1^m], [n])$

$$P_{[1^m][\bar{n}]} = \left(1 + \sum_{k=1}^{n} (-1)^k \prod_{l=1}^{k} \frac{1}{(N - m + n)} C_{(k)}\right) p_{[1^m]} \bar{p}_{[n]} \tag{5.27}$$

In appendix A.3, we give a proof of $C_{i\bar{j}} P_{R\bar{S}} = 0$ for these projectors. These expressions correctly reduce to the relevant examples from subsections 5.1–5.4

6. Counting of operators

As observed in section 3.4 Brauer elements can be used to construct multi-trace local operators from complex matrices $\Phi, \Phi^\dagger$. We also observed that the counting of these multi-trace operators is the same as counting symmetric elements in $B_N(m, n)$, which are elements that commute with the $\mathbb{C}[S_m \times S_n]$ sub-algebra of $B_N(m, n)$. A class of symmetric elements are symmetric projectors. These projectors have appeared in sections 4 and 5 as summands in central Brauer projectors. This relation between central Brauer projectors and symmetric projectors corresponds to the group theory counting of irreps of $B_N(m, n)$ in $V^{\otimes m} \otimes \bar{V}^{\otimes n}$ weighted by the multiplicity of $\mathbb{C}[S_m \times S_n]$ irreps. So the number of symmetric projectors $N_s(m, n)$ is given by

$$N_s(m, n) = \sum_{\gamma} \sum_{A} M_A^{\gamma} \tag{6.1}$$

where A labels irreps of the symmetric group $S_m \times S_n$. It is given by a pair (α, β) which is a pair of partitions of m, n respectively. Using the expression (3.13) for the multiplicities we obtain

$$N_s(m, n) = \sum_{k=0}^{min(m,n)} \sum_{\gamma_+ \vdash (m-k)} \sum_{\gamma_- \vdash (n-k)} \sum_{\alpha \vdash m} \sum_{\beta \vdash n} \left(\sum_{\delta \vdash k} g(\delta, \gamma_+; \alpha) g(\delta, \gamma_-; \beta) \right) \tag{6.2}$$

The most general symmetric element of $B_N(m, n)$ is not necessarily a projector, but we can argue that the most general element is a symmetric branching operator, a special case of which is a projector. These will be discussed in more detail in section 7. They are counted by a N_{sb}

$$\begin{aligned} N_{sb} &= \sum_{\gamma} \sum_{A} (M_A^{\gamma})^2 \\ &= \sum_{k=0}^{min(m,n)} \sum_{\gamma_+ \vdash (m-k)} \sum_{\gamma_- \vdash (n-k)} \sum_{\alpha \vdash m} \sum_{\beta \vdash n} \left(\sum_{\delta \vdash k} g(\delta, \gamma_+; \alpha) g(\delta, \gamma_-; \beta) \right)^2 \end{aligned} \tag{6.3}$$

At large N, i.e $m+n < N$ the counting of traces is equivalent to a problem of counting necklaces with coloured beads, which is solved by Polya theory. In this large N case, the counting of gauge invariant operators by Polya theory agrees with the counting in terms of Brauer algebras. When we drop the restriction and deal with finite N effects, the connection to Brauer algebras allows a simple solution of the finite N counting problem. The sums over Brauer irreps are reduced to those which appear in the decomposition of $V^{\otimes m} \otimes \bar{V}^{\otimes n}$. This will be discussed further in section 8. For other recent discussions of finite N matrix counting problems see [37–39].

The counting of traces (at large N) is given by Polya theory as

$$\begin{aligned} T(x,y) &= \prod_{n=1}^{\infty} \frac{1}{(1-x^n-y^n)} \\ &= \sum_{m=0,n=0}^{\infty} t(m,n) x^m y^n \end{aligned} \tag{6.4}$$

The coefficients $t(m,n)$ count the number of traces with m copies of Φ and n copies of $\Phi^\dagger$. For a recent discussion of this in the physics literature see [40]. We are led, from the above discussion, to

$$N_{sb}(m,n) = t(m,n) \tag{6.5}$$

There are a number of interesting cases where the multiplicities $M_A^\gamma = M_{\alpha,\beta}^\gamma$ are all either 1 or 0. In these cases $N_s(m,n) = N_{sb}(m,n) = t(m,n)$. One class of such examples is $B_N(m,1)$. In this case $t(m,1)$ can be obtained by calculating the derivative of $\frac{\partial T(x,y)}{\partial y}|_{y=0}$.

$$\begin{aligned} \frac{\partial T(x,y)}{\partial y}|_{y=0} &= \frac{1}{1-x} \prod_{n=1}^{\infty} \frac{1}{(1-x^n)} \\ &= \sum_{m_1=0}^{\infty} x^{m_1} \sum_{m_2=0}^{\infty} p(m_2) x^{m_2} \\ &= \sum_{m=0}^{\infty} x^m \sum_{k=0}^{m} p(k) \end{aligned} \tag{6.6}$$

where $p(k)$ is the number of partitions of k. Hence

$$t(m,1) = \sum_{k=0}^{m} p(k) \tag{6.7}$$

This satisfies a recursion relation

$$t(m+1,1) = p(m+1) + t(m,1) \tag{6.8}$$

The same recursion relation can be derived for $N_s(m,1)$. The sum over k for $N_s(m,1)$ has two terms. The $k=0$ term gives $p(m)p(1) = p(m)$. To get the second term, we sum over Young diagrams R of m boxes. Let such a diagram have c_1 columns of length 1, c_2

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columns of length 2 etc. In other words j corresponds to column length and c_j gives to multiplicity of that column length in the Young diagram of R. So $m = \sum_{j=0}^{m} j c_j$. For each Young diagram the factor $\sum_R \sum_{\gamma_+ \vdash (m-1)} g(\gamma_+, [1]; R)$ is equal to the number of ways of removing a box from R to get a legal Young diagram of $m-1$ boxes. This can be seen to be equal to the number of non-zero column length multiplicities c_j's. Hence,

$$N_s(m,1) = \sum_{R \vdash m} (1 + \text{ number of ways of removing a box from } R\ ) \tag{6.9}$$

Now $(1+$ number of ways of removing a box from R) is equal to number of way of adding a box to R. Hence

$$\begin{aligned} N_s(m,1) &= \sum_{R \vdash m} (1 + \text{ number of ways of removing a box from } R\ ) \\ &= \sum_{R \vdash m+1} (\text{ number of ways of removing a box from } R\ ) \end{aligned} \tag{6.10}$$

But we also know that

$$N_s(m+1,1) = p(m+1) + \sum_{R \vdash m+1} (\text{ number of ways of removing a box from } R\ ) \tag{6.11}$$

Hence $N_s(m+1,1) = p(m+1) + N_s(m,1)$. This the same recursion relation as for $t(m,1)$. It is also easily checked that $N_s(1,1) = t(1,1)$. This proves the desired identity between the number of traces and the number of Brauer irreps weighted with symmetric group decomposition multiplicities.

We have also checked for $B_N(m,2)$ in the cases $m = 1 \cdots 5$ that $N_s(m,2) = N_{sb}(m,2) = t(m,2)$. For cases such as $B_N(3,3)$, we find $t(3,3) = N_{sb}(3,3) = 38$ whereas $N_s(3,3) = 36$. We have also checked $N_{sb}(4,3) = t(4,3); N_{sb}(4,4) = t(4,4); N_{sb}(4,5) = t(4,5); N_{sb}(5,5) = t(5,5)$. Based on these non-trivial examples, and the discussion of sections 3.4, 7 we expect that (6.5) is true in general. At finite N there is a cutoff on γ in (6.3) following from (3.8) of $c_1(\gamma_+) + c_1(\gamma_-) \leq N$.

7. Orthogonal set of operators for brane-anti-brane systems.

A representation γ of $B_N(m,n)$ can be decomposed into irreducible representations A of the $\mathbb{C}[S_m \times S_n]$ sub-algebra. The index A consists of a pair (α, β) where α is a partition of m, and β is a partition of n, which is expressed as $\alpha \vdash m, \beta \vdash n$. An irrep A will generically appear with multiplicity M_A^γ, and we will use an index i which runs over this multiplicity. Let us denote by $|\gamma; A, m_A; i\rangle$ an orthonormal set of vectors in the γ representation which transforms in the i th copy of the state m_A of the irrep A of the sub-algebra. Central projectors for γ in the regular representation can be written as

$$P^\gamma = \sum_i \sum_{A, m_A} |\gamma; A, m_A; i\rangle\langle\gamma; A, m_A; i| \tag{7.1}$$

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The construction of these projectors in terms of the algebra has been discussed at length in section 4. Define

$$P^{\gamma}_{A,i} = \sum_{m_A} |\gamma; A, m_A; i\rangle\langle\gamma; A, m_A; i| \tag{7.2}$$

Here we are not summing over A, i. As we will show these commute with $\mathbb{C}[S_m \times S_n]$, but not in general with $B_N(m,n)$. Examples of these symmetric projectors have also been computed in section 5. These projectors belong to a more general class of symmetric elements.

$$Q^{\gamma}_{A,ij} = \sum_{m_A} |\gamma; A, m_A; i\rangle\langle\gamma; A, m_A; j| \tag{7.3}$$

Consider

$$\begin{aligned} hQ^{\gamma}_{A,ij}h^{-1} &= \sum_{m_A,n_A,k_A} D^A_{n_A m_A}(h)|\gamma; A, n_A; i\rangle\langle\gamma; A, k_A; j|D^A_{m_A k_A}(h^{-1}) \\ &= \sum_{n_A,k_A} D^A_{n_A k_A}(1)|\gamma; A, n_A; i\rangle\langle\gamma; A, k_A; j| \\ &= \sum_{n_A,k_A} \delta_{n_A k_A}|\gamma; A, n_A; i\rangle\langle\gamma; A, k_A; j| \\ &= Q^{\gamma}_{A,ij} \end{aligned} \tag{7.4}$$

The $D^A_{n_A m_A}(h)$ are matrix elements of h in the irrep A. This shows that $Q^{\gamma}_{A,ij}$ commutes with the subalgebra. This property is denoted by saying $Q^{\gamma}_{A,ij}$ are *symmetric branching operators*. By using an expansion of a general element of the Brauer algebra in terms of matrix elements of irreps as in [30] we expect it should be possible to prove that the symmetric branching operators provide a complete set of symmetric elements in the Brauer algebra This is supported by the counting examples we have done in section 6. Since $P^{\gamma}_{A,i} = Q^{\gamma}_{A,ii}$ we also have

$$hP^{\gamma}_{A,i}h^{-1} = P^{\gamma}_{A,i} \tag{7.5}$$

This property is expressed by saying $P^{\gamma}_{A,i}$ are symmetric projectors. Using the orthonormality of the states we can derive

$$Q^{\gamma_1}_{A,ij}Q^{\gamma_2}_{B,kl} = \delta_{\gamma_1\gamma_2}\delta_{AB}\delta_{jk}Q^{\gamma_1}_{A,il} \tag{7.6}$$

Associated with the symmetric elements Q, we can find a complete basis in the space of local operators constructed from $\Phi, \Phi^\dagger$ in four dimensional $N = 4$ SYM. It also gives a complete basis of gauge invariant operators built from the matrices $A^\dagger, B^\dagger$. We will show that this basis for operators diagonalises the correlators. We first discuss this in the context of the matrix quantum mechanics.

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7.1 Reduced 1D matrix model: orthogonal basis using Brauer

It has been shown [1, 41, 42] that the reduction of the four-dimensional action on $S^3 \times R$ leads to the Hamiltonian and SO(2) symmetry

$$\begin{aligned} H &= tr(A^\dagger A + B^\dagger B) \\ J &= tr(A^\dagger A - B^\dagger B) \end{aligned} \tag{7.7}$$

The matrices obey the algebra

$$\begin{aligned} &[A^i_j, A^{\dagger\ k}_l] = \delta^i_l \delta^k_j \\ &[B^i_j, B^{\dagger\ k}_l] = \delta^i_l \delta^k_j \\ &[A^i_j, B^{\dagger\ k}_l] = [A^i_j, B^k_l] = [A^{\dagger\ i}_j, B^{\dagger\ k}_l] = [A^{\dagger\ i}_j, B_l{}^k] = 0 \end{aligned} \tag{7.8}$$

Gauge invariant states are obtained by acting with traces of $A^\dagger$ and $B^\dagger$ on the vacuum, e.g

$$\begin{aligned} Tr(A^\dagger)^n|0> &\qquad E = J = n \\ Tr(B^\dagger)^n|0> &\qquad E = -J = n \\ Tr(A^\dagger)^n(B^\dagger)^m|0> &\qquad E = n+m, J = n-m \end{aligned} \tag{7.9}$$

Among the states obtained by acting with $A^\dagger$ a complete orthogonal set is obtained from the Schur polynomials

$$\chi_R(A^\dagger)|0> \tag{7.10}$$

They obey

$$< 0|\chi_R(A)\chi_S(A^\dagger)|0> = \delta_{RS}\frac{n!DimR}{d_R} \tag{7.11}$$

We would like to find a complete set of orthogonal states in the more general case where both $A^\dagger$ and $B^\dagger$ are acting on the vacuum.

We claim that the operators $tr_{m,n}\big(\Sigma(Q^\gamma_{A,ij})(A^\dagger \otimes B^\dagger)\big)|0>$ diagonalise the quantum mechanical inner product

$$\begin{aligned} &< 0|tr_{m,n}\big(\Sigma(Q^{\gamma_2\ \dagger}_{A_2,i_2j_2})(A\otimes B)\big)tr_{m,n}\big(\Sigma(Q^{\gamma_1}_{A_1,i_1j_1})(A^\dagger \otimes B^\dagger)\big)|0> \\ &= m!n!\delta_{\gamma_1\gamma_2}\delta_{A_1A_2}\delta_{i_1i_2}\delta_{j_1j_2}\ d_{A_1}\ Dim\gamma_1 \end{aligned} \tag{7.12}$$

Consider the l.h.s. of (7.12). We can describe it diagrammatically as in figure 8. We have used Q_1, Q_2 for $Q^{\gamma_1}_{A_1,i_1j_1}, Q^{\gamma_2\ \dagger}_{A_2,i_2j_2}$ in the figures to keep them simple. It is understood that the upper horizontal line is identified with the lower horizontal line, which expresses the identification of tensor space indices for a trace. The sum over all Wick contractions gives a sum over permutations in tensor space as in figure 9. An obvious diagrammatic manipulation, which corresponds to an identity in tensor space, results in figure 10. This allows us to write

$$
\begin{aligned}
&< 0|tr_{m,n}\big(\Sigma(Q^{\gamma_2 \dagger}_{A_2,i_2j_2})(A\otimes B)\big)tr_{m,n}\big(\Sigma(Q^{\gamma_1}_{A_1,i_1j_1})(A^\dagger\otimes B^\dagger)\big)|0>\\
&\quad = \sum_{\alpha_1\in S_m}\sum_{\alpha_2\in S_n} tr_{m,n}\big((\alpha_1\otimes\alpha_2)\Sigma(Q^{\gamma_2\ \dagger}_{A_2,i_2j_2})(\alpha_1^{-1}\otimes\alpha_2^{-1})\Sigma(Q^{\gamma_1}_{A_1,i_1j_1})\big)\\
&\quad = m!n!tr_{m,n}\big(\Sigma(Q^{\gamma_2\dagger}_{A_2,i_2j_2})\Sigma(Q^{\gamma_1}_{A_1,i_1j_1})\big)\\
&\quad = m!n!tr_{m,n}\big(Q^{\gamma_2\ \dagger}_{A_2,i_2j_2}Q^{\gamma_1}_{A_1,i_1j_1}\big)\\
&\quad = m!n!\delta_{\gamma_1\gamma_2}\delta_{A_1A_2}\delta_{i_1i_2}tr_{m,n}(Q^{\gamma_1}_{A_1,j_2j_1})\\
&\quad = m!n!\delta_{\gamma_1\gamma_2}\delta_{A_1A_2}\delta_{i_1i_2}\delta_{j_2j_1}d_{A_1}Dim\gamma_1
\end{aligned} \tag{7.13}
$$

The second line follows from the diagrammatics. The third line follows using (3.37). The fourth line follows from (3.20). In the last line we have used the Schur-Weyl duality (3.8). The factor $Dim\gamma_1$ is the dimension of the $GL(N)$ irrep labelled by γ_1 and d_{A_1} is the dimension of the corresponding irrep of $\mathbb{C}[S_m \times S_n]$. This proves (7.12).

Using the relations between the symmetric projectors $P^\gamma_{A,i}$ or central projectors P^γ in terms of these symmetric branching operators, we can derive

$$
\begin{aligned}
&< 0|tr_{m,n}(\Sigma(P^{\gamma_2}_{A_2,i_2}))(A\otimes B)tr_{m,n}(\Sigma(Q^{\gamma_1}_{A_1,i_1j_1})(A^\dagger\otimes B^\dagger))|0>\\
&\quad = m!n!\delta_{\gamma_1\gamma_2}\delta_{A_1A_2}\delta_{i_2i_1}\delta_{i_2j_1}d_{A_1}Dim\gamma_1
\end{aligned}
$$

This is proportional to $\delta_{i_1j_1}$, which guarantees that the $Q^{\gamma_1}_{A_1,i_1j_1}$ overlaps with a symmetric projector if it is actually itself a symmetric projector. Considering the overlap between operators constructed from two symmetric projectors, we have

$$
\begin{aligned}
&< 0|tr_{m,n}(\Sigma(P^{\gamma_2}_{A_2,i_2}))(A\otimes B)tr_{m,n}(\Sigma(P^{\gamma_1}_{A_1,i_1})(A^\dagger\otimes B^\dagger))|0>\\
&\quad = m!n!\delta_{\gamma_1\gamma_2}\delta_{A_1A_2}\delta_{i_1i_2}d_{A_1}Dim\gamma_1
\end{aligned} \tag{7.14}
$$

We can also show that central projectors corresponding to different irreps γ give orthogonal states:

$$
\begin{aligned}
&< 0|tr_{m,n}(\Sigma(P^{\gamma_2}))(A\otimes B)tr_{m,n}(\Sigma(P^{\gamma_1})(A^\dagger\otimes B^\dagger))|0>\\
&\quad = m!n!\delta_{\gamma_1\gamma_2}tr_{m,n}(P^{\gamma_1}) = m!n!\delta_{\gamma_1\gamma_2}\ d^{(B)}_{\gamma_1}\ Dim\gamma_1
\end{aligned} \tag{7.15}
$$

In this equation $d^{(B)}$ is a Brauer dimension. This can be derived by relating P^γ to the Q operators, or applying the diagrammatics directly to the l.h.s. of (7.15) and using the projector property $P^{\gamma_1}P^{\gamma_2} = \delta_{\gamma_1\gamma_2}P^{\gamma_1}$.

Recall from section 3 that Brauer irrep labels γ determine an integer k and partitions $\gamma_+ \vdash m-k, \gamma_- \vdash n-k$. For $k=0$, the irrep γ decomposes into a unique irrep γ_+, γ_- of $\mathbb{C}[S_m \otimes S_n]$. This means that the $k=0$ central projectors do not decompose into a sum of multiple symmetric projectors. Another special property of the $k=0$ projectors becomes apparent when we consider the field theory operators, rather than Matrix quantum mechanics operators.

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7.2 Orthogonal multi-matrix operators in 4D field theory

In the case of 4D field theory we associate gauge invariant Matrix operators, much as we do in Matrix quantum mechanics. Now we consider $: tr_{m,n}\big(\Sigma(Q^{\gamma}_{A,ij})(\Phi \otimes \Phi^{\dagger})\big) :$. The notation $: \mathcal{O} :$ for an operator $\mathcal{O}$ indicates that we have subtracted the short distance singularities, to give an operator which will have no self-contractions inside correlators as explained in the example in section 2. The orthogonality property of (7.12) has a direct analog, with identical derivation

$$
\begin{aligned}
&<: tr_{m,n}\big(\Sigma(Q^{\gamma_2\,\dagger}_{A_2,i_2j_2})(\Phi^{\dagger} \otimes \Phi)\big) : \;\; : tr_{m,n}\big(\Sigma(Q^{\gamma_1}_{A_1,i_1j_1})(\Phi \otimes \Phi^{\dagger})\big) :> \\
&\qquad = m!n!\delta_{\gamma_1\gamma_2}\delta_{A_1A_2}\delta_{i_1i_2}\delta_{j_1j_2}\; d_{A_1}\; Dim\gamma_1
\end{aligned} \tag{7.16}
$$

To be more explicit we would add the dependence on x_1, x_2 in the two operators, demonstrating their location in $\mathbb{R}^4$, and the overall factor $(x_1 - x_2)^{-2m-2n}$. We have chosen to keep the notation simple, the position dependences can be added back easily if desired. Likewise, analogously to (7.15) we have for the correlator of central projectors

$$
\begin{aligned}
&< 0| : tr_{m,n}(\Sigma(P^{\gamma_2}))(\Phi^{\dagger} \otimes \Phi) : \;\; : tr_{m,n}(\Sigma(P^{\gamma_1})(\Phi \otimes \Phi^{\dagger})) : |0 > \\
&\qquad = m!n!\delta_{\gamma_1\gamma_2} tr_{m,n}(P^{\gamma_1}) \\
&\qquad = m!n!\delta_{\gamma_1\gamma_2}\; d^{(B)}_{\gamma_1}\; Dim\gamma_1
\end{aligned} \tag{7.17}
$$

The special property of the $k = 0$ projectors is that they are orthogonal to contractions $C_{i\bar{j}}$. In the special case of $k = 0$, and denoting $\gamma_+ = R, \gamma_- = S$,

$$
C_{i\bar{j}} P_{R\bar{S}} = 0 \tag{7.18}
$$

This means that the corresponding operators have no short distance singularities

$$
: tr_{m,n}(\Sigma(P^{\gamma_1})(\Phi \otimes \Phi^{\dagger})) : \;\; = \;\; tr_{m,n}(\Sigma(P^{\gamma_1})(\Phi \otimes \Phi^{\dagger})) \tag{7.19}
$$

Any $k = 0$ irrep of Brauer determines a pair of Young diagrams (R, S) and a projector $P_{R\bar{S}}$. The nonsingular field theory operator $trP_{R\bar{S}}(\Phi \otimes \Phi^*) = tr\Sigma(P_{R\bar{S}})(\Phi \otimes \Phi^{\dagger})$ is our proposal for a giant-anti-giant operator where R determines a giant and S determines an anti-giant. These operators only exist when $c_1(R) + c_1(S) \leq N$, as is clear from (3.8). We will describe this cutoff as a non-chiral stringy exclusion principle, and we will discuss it further in section 8. In the chiral case the proposal reduces to $tr(P_R\Phi) = d_R\chi_R(\Phi)$. For $k \neq 0$ operators we expect that the subtraction will involve powers up to ϵ^{-2k} in the short distance subtraction.

7.3 Examples : $B_N(3, 1)$ and $B_N(2, 2)$

We consider here some simple examples of $B_N(m, n)$ with $m + n < N$. Take for example $m = 3$, $n = 1$. In this case there are 7 independent gauge invariant operators

$$
\begin{array}{llll}
(tr\Phi)^3 tr\Phi^{\dagger} & tr\Phi^3 tr\Phi^{\dagger} & tr\Phi^2 tr\Phi tr\Phi^{\dagger} & \\
tr\Phi^3\Phi^{\dagger} & tr\Phi^2\Phi^{\dagger} tr\Phi & tr\Phi^2 tr\Phi\Phi^{\dagger} & tr\Phi\Phi^{\dagger}(tr\Phi)^2
\end{array} \tag{7.20}
$$

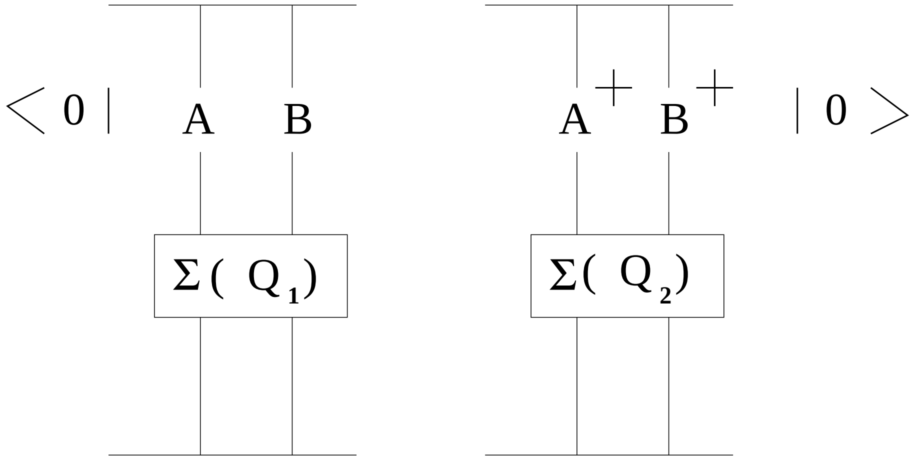

Figure 8: Diagrammatic representation of inner product

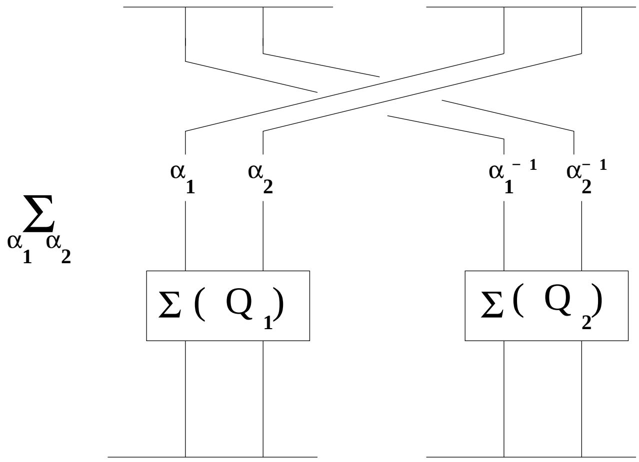

Figure 9: Sum over all possible contractions leads to a sum over permutations in tensor space

In the case of $m = 2$, $n = 2$, we have 10 gauge invariant operators

$$\begin{array}{llll} tr\Phi^2 tr(\Phi^\dagger)^2 & (tr\Phi)^2 tr(\Phi^\dagger)^2 & tr\Phi^2(tr\Phi^\dagger)^2 & (tr\Phi)^2(tr\Phi^\dagger)^2 \\ tr\Phi^2\Phi^\dagger tr\Phi^\dagger & tr\Phi tr\Phi(\Phi^\dagger)^2 & tr\Phi tr\Phi\Phi^\dagger tr\Phi^\dagger & (tr\Phi\Phi^\dagger)^2 \\ tr\Phi^2(\Phi^\dagger)^2 & tr\Phi\Phi^\dagger\Phi\Phi^\dagger & & \end{array} \tag{7.21}$$

In both cases, these listed operators do not form orthogonal bases. A complete set of orthogonal bases of gauge invariant operators is obtained by taking linear combinations of these operators, and finding such a set is solved by the use of symmetric projectors. It is explicitly shown in appendix A.4. By making the following replacement

$$\Phi \to A^\dagger, \quad \Phi^\dagger \to B^\dagger \tag{7.22}$$

we obtain a complete set of orthogonal gauge invariant states in the quantum mechanics.

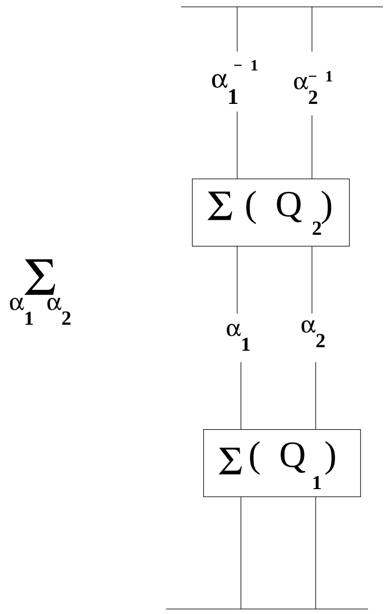

Figure 10: Tensor space identities allow the diagrammatic simplification

8. Physical interpretation of the spectrum of multi-traces

We have provided a basis of multi-trace constructed from $\Phi, \Phi^\dagger$ which diagonalise the two-point function in free 4D $N = 4$ SYM (7.16), by using the symmetric branching operators $Q^{\gamma}_{A,ij}$. By the operator-state correspondence, these give rise to orthogonal states. The symmetric branching operators also give an orthogonal basis of states in the reduced quantum mechanics of two matrices (7.13). The same combinations of multi-traces also give diagonal correlators in the zero-dimensional matrix model (2.7).

The label γ in (3.8) identifies simultaneously the irreps of $\mathrm{U}(N)$ and $B_N(m,n)$ which appear in the decomposition of $V^{\otimes m} \otimes \bar{V}^{\otimes n}$. In the Young diagram description of $\mathrm{U}(N)$ negative row lengths are allowed. The set of positive rows defines γ_+ and the set of negative rows define γ_-. For example with $m = 6, n = 4, N = 6$ a possible γ is $\gamma = [3, 2, 1, -1, -1, -2]$. This determines $\gamma_+ = [3, 2, 1]$, $\gamma_- = [2, 1, 1]$ and $k = 0$. An example such as $\gamma = [2, 1, 1, 1, -1, -2]$ determines $k = 1, \gamma_+ = [2, 1, 1, 1], \gamma_- = [2, 1]$. Given a pair of Young diagrams R, S with m, n boxes respectively, and $c_1(R) + c_1(S) \leq N$, there is always a γ with $k = 0$

$$\begin{aligned} \gamma_i &= r_i && \text{for } i = 1 \cdots c_1 \\ \gamma_i &= 0 && \text{for } i = c_1 + 1 \ldots N - \bar{c}_1 \\ \gamma_i &= -s_{N-i+1} && \text{for } i = N - \bar{c}_1 + 1 \cdots N \end{aligned} \tag{8.1}$$

This γ corresponds to $k = 0, \gamma_+ = R, \gamma_- = S$. For such a γ, the $Q^{\gamma}_{A,ij}$ reduces to a single central projector, which we have called $P_{R\bar{S}}$. The multi-trace operator associated to this projector is our proposal for the ground state of the brane-anti-brane system made as a composite of the brane described by R and the anti-brane described by S. The $k = 0$

projectors are annihilated by the contraction operators in $B_N(m,n)$ as a result they have no short distance singularities. As explained in section 2, the Wick-contractions between Φ and Φ^* result in a contraction in tensor space. When we construct a composite operator by composing a projector $P_{R\bar{S}}$ with $\Phi \otimes \Phi^*$ in $V^{\otimes m} \otimes \bar{V}^{\otimes n}$ i.e $tr_{m,n}(P_{R\bar{S}}\Phi \otimes \Phi^*)$, and we consider the short distance subtractions we encounter precisely the products $C_{i\bar{j}}P_{R\bar{S}} = 0$. Hence the short-distance singularities in $tr_{m,n}(P_{R\bar{S}}\Phi \otimes \Phi^*)$ vanish without the need for subtractions. Equivalently

$$: \; tr_{m,n}(P_{R\bar{S}}\Phi \otimes \Phi^*) \; : \; = \; tr_{m,n}(P_{R\bar{S}}\Phi \otimes \Phi^*) \tag{8.2}$$

The construction of a composite γ from the pair R, S as in (8.1) is nothing but the addition of the $\mathrm{U}(N)$ weights associated with R and $\bar{S}$, which also appears in 2dYM. The $\mathrm{U}(N)$ weights are related to generalised spacetime charges connected to the integrability of the theory [44] and from this point of view it may be possible to develop a spacetime interpretation of the addition of weights for the composite system.

The critical reader might object that we should not expect an exact eigenstate of the string theory Hamiltonian corresponding to a brane-anti-brane configuration which should be unstable due to tachyon condensation. However, at zero Yang-Mills coupling, or the tensionless string limit, the tachyon formally becomes massless. So we should expect a map between brane-anti-brane configurations and exact eigenstates to be possible. Even in the opposite limit of semiclassical gravity, with a probe description of giants and anti-giants following [2] we might expect that when the brane and anti-brane are far from each other there should be at least an approximate eigenstate corresponding to their composite. In fact the description of static non-supersymmetric supergravity solutions in terms of brane-anti-brane parameters (for a recent discussion and references to the relevant literature see [43]) suggests that it might also be possible to extrapolate an appropriate brane-anti-brane description of the eigenstates of the free Yang-Mills Hamiltonian to the supergravity regime, where back-reaction on spacetime would produce a non-supersymmetric generalization of the solutions of LLM [12].

The multi-traces constructed from $P_{R\bar{S}}$ thus give us the lowest energy state we can associate to the composite of brane R and anti-brane S. For the more general Q operators, associated with $\gamma(k \geq 1)$, a natural proposal is that for $\gamma = (k, \gamma_+ \vdash m-k, \gamma_- \vdash n-k)$ we have states of energy $m+n$, which are excitations of the brane-anti-brane system (γ_+, γ_-). The excitations carry $2k$ units of energy and have a multiplicity

$$\sum_{R \vdash m, S \vdash n} (M^{\gamma}_{RS})^2 \tag{8.3}$$

This multiplicity and the form of the corresponding operators $Q^{\gamma}_{(RS),ij}$ suggest that they should also be interpreted as states arising from a descent procedure from the brane-anti-brane pair R, S of energies m, n. The descent procedure in question involves partons (constituents of the brane each carrying a unit of angular momentum each) and anti-partons (constituents of the anti-brane each carrying a unit of angular momentum). The integer k can be interpreted as the number of partons and anti-partons, from the brane-anti-brane pair R, S combining to form a stringy excitation with $2k$ units of energy. For fixed initial brane-configuration R, S and final brane configuration (γ_+, γ_-) the multiplicity $(M^{\gamma}_{RS})^2$ and the matrix structure of the associated operators $Q^{\gamma}_{(RS),ij}$ (7.6) is suggestive of a Chan-Paton interpretation of the i, j indices labelling the stringy excitations. It would be very interesting to construct a dynamical model of the stringy excitations and the descent procedure which precisely accounts for the multiplicity M^{γ}_{RS} known in terms of Littlewood-Richardson coefficients (3.13). A useful analogy might be the role of the elementary field Φ as a parton for long strings in the BMN limit [35], which is developed in terms of a concrete string bit model in [45].

The description of brane-anti-branes in terms of irreps γ of the Brauer algebras $B_N(m, n)$ or $\mathrm{U}(N)$ reveals an interesting finite N cutoff on the brane-anti-brane configurations. Let us go back to the example of $m = 6, n = 4, N = 6$. Since γ has exactly 6 rows, no choice of γ will give a pair such as $\gamma_+ = [2, 2, 1, 1], \gamma_- = [2, 1, 1]$. In general we have the bound

$$c_1(\gamma_+) + c_1(\gamma_-) \leq N \tag{8.4}$$

We will call this the *non-chiral stringy exclusion principle*, following the terminology of stringy exclusion principle for the cutoff in the holomorphic case [46]. The bound on the individual branes or anti-branes $c_1(\gamma_+) \leq N$ and $c_1(\gamma_-) \leq N$ can be understood in the semiclassical probe picture [2] or the supergravity picture [12] and a more speculative explanation of related cutoffs on single traces in terms of non-commutative spacetime was explored [47]. An analogous spacetime understanding of the the new bound on composites is a fascinating challenge for the spacetime picture. If we are given a pair $\gamma_+ = R, \gamma_- = S$ violating the bound we can still form a $\mathrm{U}(N)$ irrep γ by adding the corresponding highest weights as follows

$$\begin{aligned} &\gamma_i = r_i && \text{for } i = 1 \cdots c_1 - E \\ &\gamma_i = r_i - s_{N-i+1} && \text{for } \ i = c_1 - E + 1 \cdots c_1 \\ &\gamma_i = -s_{N-i+1} && \text{for } \ i = c_1 + 1 \cdots N \end{aligned} \tag{8.5}$$

It is easy to check that $\gamma_i \geq \gamma_{i+1}$ as required. This γ has $k = \sum_{i=c_1-E+1}^{c_1} min(r_i, s_{N-i+1})$, so that $2k$ is the number of boxes removed after superposing the Young diagram of R with that of $\bar{S}$ to get the γ. Following the interpretation above, this says that when the bound is violated, the composite brane-anti-brane system is actually a stringy excited state of a brane-anti-brane pair (γ_+, γ_-) determined by (8.5) with energy $m + n - 2k = m + n - 2\sum_{i=c_1-E+1}^{c_1} min(r_i, s_{N-i+1})$, which has been excited by a stringy excitation with $2k$ units of energy.

We will now describe some technical consequences of the non-chiral stringy exclusion principle. Since pairs $\gamma = (k = 0, R \vdash m, S \vdash n)$ do not appear in (3.8) when $c_1(R)+c_1(S) > N$, we may ask what happens to the general formulae for $P_{R\bar{S}}$ we wrote in this case. It turns out that in some cases, the projector becomes ill-defined with 0 appearing in the denominator. In other cases, it can be seen that the projector acting on $V^{\otimes m} \otimes \bar{V}^{\otimes n}$ gives zero. This means that $tr_{m,n}(P_{R\bar{S}} \Phi \otimes \Phi^*)$ vanishes. Since the first term in $P_{R\bar{S}}$ is $p_R p_S$, this means that the vanishing leads to a matrix identity for $\chi_R(\Phi)\chi_S(\Phi^\dagger)$ in terms of multi-traces where $\Phi, \Phi^\dagger$ appear in the same trace.

Before getting to a non-chiral example we review an analog in the chiral case. Matrix identities follow from the fact that the n-fold antisymmetiser acting on the N-dimensional space V vanishes when $n > N$.

$$p_{[1^n]} V^{\otimes n} = 0 \quad (n > N) \tag{8.6}$$

Let us consider an $N = 2$ matrix ϕ. In this case, we have the following identity

$$tr(\phi^3) = \frac{3}{2}(tr\phi)tr(\phi^2) - \frac{1}{2}(tr\phi)^3 \tag{8.7}$$

which is a direct consequence of (8.6) because (8.7) can be rewritten as

$$tr_3(p_{[1^3]}\phi) = \frac{1}{6}\left((tr\phi)^3 - 3(tr\phi)tr(\phi^2) + 2tr(\phi^3)\right) = 0 \tag{8.8}$$

Now we consider a non-chiral example of matrix identities following from the vanishing of projectors. For example, we can show

$$P_{[1^2][\bar{1}]} V^{\otimes 2} \otimes \bar{V} = 0 \quad (N = 2) \tag{8.9}$$

as follows. For a state $w = v_1 \otimes v_2 \otimes \bar{v}_1 \in V^{\otimes 2} \otimes \bar{V}$, we have

$$p_{[1^2]} w = v_{[1} \otimes v_{2]} \otimes \bar{v} \tag{8.10}$$

and

$$\begin{aligned} p_{[1^2]} C w &= \sum_{k=1}^{2} v_{[k} \otimes v_{2]} \otimes \bar{v}_k \\ &= v_{[1} \otimes v_{2]} \otimes \bar{v}_1 \end{aligned} \tag{8.11}$$

Hence using (5.12), with $N = 2$ we obtain

$$P_{[1^2][\bar{1}]} w = (1 - C)\, p_{[1^2]} w = 0 \tag{8.12}$$

From this equation, for two $N = 2$ matrices A, B, we have

$$tr_{2,1}(P_{[1^2][\bar{1}]} A \otimes A \otimes B^T) = tr_{2,1}(\Sigma(P_{[1^2][\bar{1}]}) A \otimes A \otimes B) = 0 \tag{8.13}$$

This gives the following matrix identity,

$$tr_2(p_{[1^2]}A)tr(B) = tr(A)tr(AB) - tr(A^2 B) \tag{8.14}$$

where

$$tr_2(p_{[1^2]}A) = \frac{1}{2}(trA)^2 - \frac{1}{2}tr(A^2). \tag{8.15}$$

For two $N = 3$ matrices A and B, we have the following identity

$$\begin{aligned} tr_2(p_{[1^2]}A)tr_2(p_{[1^2]}B) &= trAtrBtrAB - trA^2BtrB - trAtrAB^2 + trA^2B^2 \\ &\quad - \frac{1}{2}trABtrAB + \frac{1}{2}trABAB \end{aligned} \tag{8.16}$$

This equation comes from

$$tr_{2,2}(P_{[1^2][\bar{1}^2]} A^{\otimes 2} \otimes B^{T\ \otimes 2}) = tr_{2,2}(\Sigma(P_{[1^2][\bar{1}^2]}) A^{\otimes 2} \otimes B^{\otimes 2}) = 0 \tag{8.17}$$

which is a consequence of

$$P_{[1^2][\bar{1}^2]} V^{\otimes 2} \otimes \bar{V}^{\otimes 2} = 0 \quad (N = 3) \tag{8.18}$$

The skeptical reader might wonder if we can bypass the nonchiral exclusion principle (8.4) simply by proposing : $\chi_R(\Phi)\chi_S(\Phi^\dagger)$: as a dual. The first objection to this is that such a proposal does not belong to a diagonal basis. The $P_{R\bar{S}}$ are a special case of the complete orthogonal set of symmetric branching operators of section 7. But a more startling failure of the naive proposal is the fact explained above, that at finite N, the products of characters become equal to sums of multi-traces where products $\Phi\Phi^\dagger$ appear within the same trace.

9. Summary and discussion

For any Young diagram R, there is an operator $\chi_R(\Phi)$ and a spherical D-brane giant graviton. For any Young diagram S, there is a spherical $\bar{D}3$-brane giant graviton and corresponding operator $\chi_S(\Phi^\dagger)$. $\chi_R(\Phi)$ is a holomorphic continuation of characters $\chi_R(U)$ of the unitary group. We can view $\chi_S(\Phi^\dagger)$ is a continuation of $\chi_S(U^\dagger)$. Associated with a pair of Young diagrams, are "coupled representations" of $\mathrm{U}(N)$ which play an important role in 2dYM. The coupled character is obtained from the trace in $V^{\otimes m} \otimes \bar{V}^{\otimes n}$ of a projector (4.1). These projectors are constructed from Brauer algebras $B_N(m,n)$. These same projectors can be used to construct composite operators involving the complex matrix Φ and its conjugate Φ^*. We have given a number of useful general formulae for these operators in section 4, and discussed several examples in section 5 and the appendices. These operators are proposed as candidate gauge theory duals for brane-anti-brane systems. They exist when $c_1(R) + c_1(S) \leq N$. They have an interesting property that they are unchanged by short distance subtractions, generalising the simple example discussed in section 2. We also described the complete set of operators that can be constructed from $\Phi, \Phi^\dagger$. They are related to symmetric elements of $B_N(m,n)$, i.e those invariant under conjugation by the subalgebra $S_m \times S_n$. We gave a group theoretic counting of these operators and described corresponding branching operators which lead to an orthogonal basis for the two point functions.

Our calculations have been done in the zero coupling limit $g_{YM}^2 = 0$. In the case of BPS objects, these results can be extrapolated to strong coupling using non-renormalisation theorems, where they apply to the weakly coupled gravity regime. In this non-supersymmetric set-up, important physics should be contained in the mixing with fields other than $\Phi, \Phi^\dagger$ which occurs when the coupling is turned on. The basis of operators described here should be a natural starting point for perturbation away from zero coupling. It will be interesting to apply the technology for constructing operators corresponding to strings between branes, developed in [48–50] to study the strings between brane and anti-brane using our proposed brane-anti-brane operators $tr_{m,n}(P_{R\bar{S}}\Phi \otimes \Phi^*)$. This should shed light on tachyon condensation (see [51, 52] for reviews) in AdS from the dual gauge theory point of view, in regimes not accessible to perturbative string theory.

We believe we have made a convincing case that brane-anti-brane giant graviton systems at zero coupling are dual to gauge invariant operators constructed from $k = 0$ projectors $P_{R\bar{S}}$ (which belong to a complete orthogonal family of branching operators $Q^{\gamma}_{A,i,j}$ outlined in section 7.2). Further work can be done to give more explicit formulae for the branching operators. However further independent tests, beyond orthogonality and completeness, can be considered. If we can find a regime at strong 't Hooft coupling where the energy of giant-anti-giant gravitons is calculable in the space-time picture, we can conduct comparisons between the dimensions of our proposed operators and the strong coupling results. For example if the angular momenta of the giant-anti-giant pair are such that their separation is large enough for effects of tachyons to be neglected, then the ground state energy of the pair will be simply the sum of individual brane and anti-brane energies. Since the dimension of our proposed composites at zero coupling is also a sum, a consistency check on our proposal would be that this dimension flows to the same value at strong coupling. The simplest way this can happen is if the dimension of the composite is not renormalised. Such a scenario can be checked by 1-loop computations. In any case, the 1-loop anomalous dimensions will be of interest in further exploring the duality between composites of holo-antiholomorphic operators and brane-antibrane systems.

We have related the counting of multi-trace operators to Brauer algebras and using this interpretation of the counting we have proposed a brane-anti-brane interpretation of the operators (section 8). We uncovered a non-chiral stringy exclusion principle (8.4): a cut-off on brane-anti-brane states which is stronger than the cutoff on the individual branes and anti-branes. These general lessons on the counting and finite N effects should be relevant at strong coupling.

The AdS/CFT set-up thus has allowed the unique opportunity to find exact quantum operators corresponding to brane-anti-branes. It is natural to ask if there are lessons here for brane-anti-brane physics more generally. Brane-anti-brane degrees of freedom are used in black-hole counting. It has been generally a confusing issue, whether we can expect these unstable configurations to correspond to exact eigenstates of a String Theory Hamiltonian. The lesson here is that we can certainly expect exact eigenstates because we are taking a limit of zero tension, where the tachyon becomes massless. In this limit we have been able to construct exact operators for brane-anti-brane systems using Brauer algebras, and classified the stringy excitations of the brane-anti-brane systems using these algebras. It will be interesting to see how far one can extend this discussion to more general backgrounds and to the counting of the degrees of freedom of stringy non-supersymmetric systems such as those considered in [53].

One of our main objects of interest in this paper has been about projectors in the Brauer algebra. Another algebraic structure which captures many properties of tachyon condensation is K-theory [54–56]. The algebraic version of K-theory involves equivalence classes of projectors. It is interesting to ask if the Brauer algebras, perhaps in an inductive limit of $m, n \to \infty$, and their connections to K-theory, might provide an algebraic structure which is relevant to brane-anti-brane systems in a general background.

JHEP11(2007)078

Acknowledgments

We thank Tom Brown, Paul Heslop, Hally Ingram, Horatiu Nastase, Costis Papageorgakis, Gabriele Travaglini for useful discussions. SR is supported by a STFC Advanced Fellowship and in part by the EC Marie Curie Research Training Network MRTN-CT-2004-512194. YK is supported by STFC grant PP/D507323/1 “String theory, gauge theory and gravity”. He thanks theoretical physics laboratory at RIKEN for hospitality while part of this work was done and Yuuichirou Shibusa for useful discussions. He also thanks the Yukawa Institute for Theoretical Physics at Kyoto University. Discussions during the YITP workshop YITP-W-07-05 on “String Theory and Quantum Field Theory” were useful to complete this work.

A. Calculations in Brauer algebras and applications to gauge theory operators

A.1 Calculation of duals of Brauer elements

In this section, we consider the duals of Brauer elements. In particular we use (4.8) to find explicit formulae for the dual of the identity 1^* for some examples. We use this to obtain projectors for $k = 0$ representations using the formula (4.7). In the calculation of 1^*, we need values of the character of the symmetric group. They are listed in [34], for example.

A.1.1 Duals for $B_N(1,1)$

$B_N(1,1)$ can be mapped to $\mathbb{C}[S_2]$. The Ω_2 factor in S_2 is given by $\Omega_2 = 1 + s_1/N$, and the inverse of it is

$$\Omega_2^{-1} = \frac{N^2}{N^2-1}\left(1 - \frac{s_1}{N}\right) \tag{A.1}$$

Because the inverse map Σ^{-1} of s_1 is given by $C_{1\bar{1}}$, 1^* can be calculated as

$$1^* = \frac{1}{N^2-1}\left(1 - \frac{C_{1\bar{1}}}{N}\right) \tag{A.2}$$

Then using

$$t_{[1][\bar{1}]} = N^2 - 1 \tag{A.3}$$

we get a $k = 0$ projector

$$P_{[1][1]} = t_\lambda 1^* = 1 - \frac{C_{1\bar{1}}}{N} \tag{A.4}$$

which can be easily checked to satisfy $(P_{[1][1]})^2 = P_{[1][1]}$ using $(C_{1\bar{1}})^2 = NC_{1\bar{1}}$.

A.1.2 Duals for $B_N(2,1)$

In this case, 1^* is given by

$$\frac{1}{(3!)^2}\sum_T \frac{d_T^2}{dimT}\sum_{\sigma\in S_3}\chi_T(\sigma^{-1})\Sigma^{-1}(\sigma) \tag{A.5}$$

where T is an irrep of S_3. The relationship between $B_N(2,1)$ and $\mathbb{C}[S_3]$ is

$$\begin{aligned}\Sigma^{-1}(T_{[2,1]}) &= s_1 + C \\ \Sigma^{-1}(T_{[3]}) &= Cs_1\end{aligned} \tag{A.6}$$

where T_r is the sum of elements belonging to a conjugacy class r which is labelled by the cycle structure. Using the values of the character[1]

$$\begin{array}{lll}\chi_{[3]}(1) = d_{[3]} = 1 & \chi_{[3]}(s_1) = 1 & \chi_{[3]}(s_1s_2) = 1 \\ \chi_{[1^3]}(1) = d_{[1^3]} = 1 & \chi_{[1^3]}(s_1) = -1 & \chi_{[1^3]}(s_1s_2) = 1 \\ \chi_{[2,1]}(1) = d_{[2,1]} = 2 & \chi_{[2,1]}(s_1) = 0 & \chi_{[2,1]}(s_1s_2) = -1\end{array} \tag{A.7}$$

1^* can be calculated as

$$1^* = \frac{1}{N(N-s)(N+2s)}\left(1 - \frac{1}{N\Omega_2}C\right) \tag{A.8}$$

Using

$$\begin{aligned}t_{[2][\bar{1}]} &= \frac{1}{2}N(N-1)(N+2) \\ t_{[1,1][\bar{1}]} &= \frac{1}{2}N(N+1)(N-2)\end{aligned} \tag{A.9}$$

we obtain two $k=0$ projectors corresponding to $R = [2]$ or $[1^2]$

$$\begin{aligned}P_{R[\bar{1}]} &= t_{R[\bar{1}]}1^*\frac{m!}{d_R}p_R \\ &= t_{R[\bar{1}]}\frac{1}{N}\frac{1}{(N-s)(N+2s)}\left(1 - \frac{1}{N\Omega_2}C\right)2p_R \\ &= \left(1 - \frac{1}{N\Omega_2}C\right)p_R\end{aligned} \tag{A.10}$$

where we have used $sp_{[2]} = p_{[2]}$ and $sp_{[1,1]} = -p_{[1,1]}$.

A.1.3 Duals for $B_N(2,2)$

In this case, 1^* is given by

$$\frac{1}{(4!)^2}\sum_T \frac{d_T^2}{dimT}\sum_{\sigma\in S_4}\chi_T(\sigma^{-1})\Sigma^{-1}(\sigma) \tag{A.11}$$

where T is an irrep of S_4. After some calculations using the values of the character and the mapping rule

$$\begin{aligned}\Sigma^{-1}(T_{[2,1^2]}) &= C + s + \bar{s} \\ \Sigma^{-1}(T_{[3,1]}) &= C(s+\bar{s}) \\ \Sigma^{-1}(T_{[4]}) &= Cs\bar{s} + C^{(2)}s \\ \Sigma^{-1}(T_{[2,2]}) &= C^{(2)} + s\bar{s}\end{aligned} \tag{A.12}$$

[1] Note that $\chi_R(g) = \chi_R(g')$ when g and g' are conjugate each other.

we get the following expression

$$
\begin{aligned}
1^* &= \frac{1}{N^2(N^2-1)(N^2-4)(N^2-9)}\times \\
&\Big(N^4 - 8N^2 + 6 - N(N^2-4)\Sigma^{-1}(T_{[2,1^2]}) + (2N^2-3)\Sigma^{-1}(T_{[3,1]}) \\
&-5N\Sigma^{-1}(T_{[4]}) + (N^2+6)\Sigma^{-1}(T_{[2,2]})\Big) \qquad\qquad (A.13)\\
&= \frac{1}{N^2(N^2-1)(N^2-4)(N^2-9)}\times \\
&\Big(N^4 - 8N^2 + 6 - N(N^2-4)(C + s + \bar{s}) + (2N^2-3)C(s+\bar{s}) \\
&-5N(Cs\bar{s} + C^{(2)}s) + (N^2+6)(C^{(2)} + s\bar{s})\Big) \qquad\qquad (A.14)
\end{aligned}
$$

where $C = C_{1\bar{1}} + C_{1\bar{2}} + C_{2\bar{1}} + C_{2\bar{2}}$ and $C_{(2)} = C_{1\bar{1}}C_{2\bar{2}} + C_{1\bar{2}}C_{2\bar{1}}$.

We consider $(R,S) = ([2],[2])$. The use of $sp_{[2]} = p_{[2]}$ and $\bar{s}\bar{p}_{[2]} = \bar{p}_{[2]}$ simplifies the expression of $1^*p_{[2]}\bar{p}_{[2]}$ as

$$
1^*p_{[2]}\bar{p}_{[2]} = \frac{1}{N^2(N-1)(N+3)}\left(1 - \frac{1}{N+2}C + \frac{1}{(N+1)(N+2)}C_{(2)}\right) \qquad (A.15)
$$

Using

$$
t_{[2][\bar{2}]} = \frac{1}{4}N^2(N-1)(N+3) \qquad (A.16)
$$

we obtain a projector corresponding to $(R,S) = ([2],[2])$

$$
P_{[2][\bar{1}]} = \left(1 - \frac{1}{N+2}C + \frac{1}{(N+1)(N+2)}C_{(2)}\right)p_{[2]}\bar{p}_{[2]} \qquad (A.17)
$$

Other cases $(R,S) = ([1^2],[1^2]), ([2],[1^2]), ([1^2],[2]])$ can be done similarly to obtain (5.17).

A.1.4 Duals for $B_N(3,1)$

Because $B_N(3,1)$ can also be mapped to $\mathbb{C}[S_4]$, we can use the equation (A.13). Though both of these two cases $B_N(2,2)$ and $B_N(3,1)$ can be mapped to the same group algebra $\mathbb{C}[S_4]$, the mapping rule is different, and the mapping rule in this case is given by

$$
\begin{aligned}
\Sigma^{-1}(T_{[2,1^2]}) &= C + T_{[2,1]} \\
\Sigma^{-1}(T_{[3,1]}) &= CT_{[2,1]} - D + T_{[3]} \\
\Sigma^{-1}(T_{[4]}) &= CT_{[3]} \\
\Sigma^{-1}(T_{[2,2]}) &= D \qquad\qquad (A.18)
\end{aligned}
$$

where $C = C_{1\bar{1}} + C_{2\bar{1}} + C_{3\bar{1}}$, $D = s_2C_{1\bar{1}} + s_1s_2s_1C_{2\bar{1}} + s_1C_{3\bar{1}}$. Note that T in the l.h.s. of (A.18) is an element of S_4 while T in the r.h.s. is an element of S_3. Then we obtain

$$
\begin{aligned}
1^* &= \frac{1}{N^2(N^2-1)(N^2-4)(N^2-9)}\times \\
&\Big(N^4 - 8N^2 + 6 - N(N^2-4)(C + T_{[2,1]}) + (2N^2-3)(CT_{[2,1]} - D + T_{[3]}) \\
&-5N(CT_{[3]}) + (N^2+6)(s_2C_{1\bar{1}} + s_1s_2s_1C_{2\bar{1}} + s_1C_{3\bar{1}})\Big) \qquad\qquad (A.19)
\end{aligned}
$$

Note that T is a central element, and using $T_r p_R = (\chi_R(T_r)/d_R)p_R$, we get

$$\begin{aligned}
1^* p_{[3]} &= \frac{1}{N(N^2-1)(N+3)}\left(1 - \frac{1}{N+2}C\right) \\
1^* p_{[1^3]} &= \frac{1}{N(N^2-1)(N-3)}\left(1 - \frac{1}{N-2}C\right) \\
1^* p_{[2.1]} &= \frac{1}{N^2(N^2-4)}\left(1 - \frac{N}{N^2-1}\Omega_2^{<i>}C_{i\bar{1}}\right)p_{[2.1]}
\end{aligned} \tag{A.20}$$

where we have used $s_i p_{[3]} = p_{[3]}$, $s_i p_{[1^3]} = -p_{[1^3]}$ and the values of the character (A.7). Using

$$\begin{aligned}
t_{[3][\bar{1}]} &= \frac{1}{6}N(N^2-1)(N+3) \\
t_{[1^3][\bar{1}]} &= \frac{1}{6}N(N^2-1)(N-3) \\
t_{[2,1][\bar{1}]} &= \frac{1}{3}N^2(N^2-1)(N+3)
\end{aligned} \tag{A.21}$$

the three equations (A.20) are unified to be

$$1^* p_R = \frac{d_R}{3!t_{R\bar{S}}}\left(1 - \frac{1}{N\Omega_3}\Omega_2^{<i>}C_{i\bar{1}}\right)p_R \tag{A.22}$$

and we reproduce the expression of projectors (5.15).

A.2 Algebra relations in $B_N(3,1)$ and $B_N(2,2)$

In this section, we list some useful formulae for $B_N(3,1)$ and $B_N(2,2)$ which are used in the construction of projectors in sections 5.3 and 5.4.

A.2.1 $B_N(3,1)$

$$\begin{aligned}
C^2 &= NC + T_{[2,1]}C - D \\
D^2 &= NC + T_{[2,1]}C - D \\
CD &= ND + T_{[3]}C
\end{aligned} \tag{A.23}$$

where $T_{[2,1]} = s_1 + s_2 + s_1 s_2 s_1$ and $T_{[3]} = s_1 s_2 + s_2 s_1$ are central elements in S_3.

$$(T_{[3]})^2 = T_{[3]} + 2, \quad (T_{[2,1]})^2 = 3(T_{[3]} + 1), \quad T_{[2,1]}T_{[3]} = 2T_{[2,1]} \tag{A.24}$$

$$\begin{aligned}
T_{[2,1]}D &= C + T_{[3]}C \\
T_{[3]}D &= T_{[2,1]}C - D
\end{aligned} \tag{A.25}$$

A.2.2 $B_N(2,2)$

$$\begin{aligned}
C^2 &= (N + s + \bar{s})C + 2C_{(2)} \\
CC_{(2)} = C_{(2)}C &= C_{(2)}(2N + s + \bar{s}) \\
(C_{(2)})^2 &= C_{(2)}N(N+s) \\
C_{(2)}s &= C_{(2)}\bar{s}
\end{aligned} \tag{A.26}$$

A.3 Proof of $C_{i\bar{j}}P_{R\bar{S}} = 0$ for composites of symmetric and antisymmetric representations

In this section, we give a proof of $C_{i\bar{j}}P_{R\bar{S}} = 0$ for $R = [m]$ or $[1^m]$ and $S = [n]$ or $[1^n]$.

We first consider $(R, S) = ([m], [n])$. We recall the expression of the projector in this case

$$
\begin{aligned}
P_{[m][\bar{n}]} &= \left(1 - \frac{1}{N+m+n-2}C_{(1)} \right.\\
&\qquad \left. + \frac{1}{(N+m+n-3)(N+m+n-2)}C_{(2)} + \cdots \right) p_{[m]}\bar{p}_{[n]} \\
&= \left(1 + \sum_{k=1}^{n}(-1)^k \prod_{l=1}^{k} \frac{1}{(N+m+n-l-1)} C_{(k)}\right) p_{[m]}\bar{p}_{[n]}
\end{aligned} \tag{A.27}
$$

where

$$
\begin{aligned}
C_{(1)} &= \sum_{ij} C_{i\bar{j}} \quad C_{(2)} = \frac{1}{2}\sum_{i\neq k}\sum_{j\neq l} C_{i\bar{k}}C_{j\bar{l}} \quad \cdots \\
C_{(k)} &= \frac{1}{k!}\sum_{i_a\neq i_b}\sum_{j_a\neq j_b} C_{i_1\bar{j}_1}C_{i_2\bar{j}_2}\cdots C_{i_k\bar{j}_k}
\end{aligned} \tag{A.28}
$$

In order to show $C_{1\bar{1}}P_{[m][n]} = 0$, we need to evaluate $C_{1\bar{1}}C_{(k)}$ acting on $p_{[m]}\bar{p}_{[n]}$. The $k = 1$ case is calculated as

$$
\begin{aligned}
C_{1\bar{1}}C_{(1)}p_{[m]}\bar{p}_{[n]} &= \left(NC_{1\bar{1}} + C_{1\bar{1}}\sum_{k\neq 1}(1k) + C_{1\bar{1}}\sum_{l\neq 1}(\bar{1}\bar{l}) + C_{1\bar{1}}\sum_{l\neq 1, k\neq 1} C_{k\bar{l}}\right) p_{[m]}\bar{p}_{[n]} \\
&= \left(C_{1\bar{1}}(N+m+n-2) + C_{1\bar{1}}C_{(1)}^{<1,\bar{1}>}\right) p_{[m]}\bar{p}_{[n]}
\end{aligned} \tag{A.29}
$$

where we have defined

$$
C_{(1)}^{<1,\bar{1}>} = \sum_{k\neq 1, l\neq 1} C_{k\bar{l}} \tag{A.30}
$$

which is the sum of single contractions avoiding $\{1, \bar{1}\}$. To show the second line, we have used

$$
\sigma p_{[m]} = p_{[m]} \tag{A.31}
$$

for any transposition $\sigma \in S_m$. We next define

$$
C_{(k)}^{<1,\bar{1}>} = \frac{1}{k!}\sum_{i_a\neq i_b\neq 1}\sum_{j_a\neq j_b\neq 1} C_{i_1\bar{j}_1}C_{i_2\bar{j}_2}\cdots C_{i_k\bar{j}_k} \tag{A.32}
$$

for any k, which is the sum of k disjoint contractions avoiding $\{1, \bar{1}\}$. Then we have

$$
\begin{aligned}
C_{1\bar{1}}C_{(2)}p_{[m]}\bar{p}_{[n]} &= \left((N+m+n-3)C_{1\bar{1}}C_{(1)}^{<1,\bar{1}>} + C_{1\bar{1}}C_{(2)}^{<1,\bar{1}>}\right) p_{[m]}\bar{p}_{[n]} \\
C_{1\bar{1}}C_{(k)}p_{[m]}\bar{p}_{[n]} &= \left((N+m+n-k-1)C_{1\bar{1}}C_{(k-1)}^{<1,\bar{1}>} + C_{1\bar{1}}C_{(k)}^{<1,\bar{1}>}\right) p_{[m]}\bar{p}_{[n]} \\
C_{1\bar{1}}C_{(n)}p_{[m]}\bar{p}_{[n]} &= (N+m+n-n-1)C_{1\bar{1}}C_{(n-1)}^{<1,\bar{1}>} p_{[m]}\bar{p}_{[n]}
\end{aligned} \tag{A.33}
$$

Using these equations, we can show

$$
\begin{aligned}
C_{1\bar{1}}P_{[m][\bar{n}]} &= \left(C_{1\bar{1}} + \sum_{k=1}^{n}(-1)^k \prod_{l=1}^{k} \frac{C_{1\bar{1}}C_{(k)}}{(N+m+n-l-1)}\right) p_{[m]}\bar{p}_{[n]} \\
&= \left(C_{1\bar{1}} + \sum_{k=1}^{n}(-1)^k \left(f^{(k)} + f^{(k+1)}\right)\right) p_{[m]}\bar{p}_{[n]} \\
&= 0
\end{aligned}
\tag{A.34}
$$

where

$$
f^{(k)} \equiv \prod_{l=1}^{k-1} \frac{C_{1\bar{1}}C^{<1,\bar{1}>}_{(k-1)}}{(N+m+n-l-1)} \quad (k=2,\cdots,n) \quad f^{(1)} \equiv C_{1\bar{1}} \quad f^{(n+1)} = 0 \tag{A.35}
$$

In the same way, we can give a proof $C_{i\bar{j}}P_{R\bar{S}} = 0$ for other cases. For antisymmetric representation, (A.31) gets replaced with

$$
\sigma p_{[1^m]} = -p_{[1^m]} \tag{A.36}
$$

for any transposition σ.

A.4 Orthogonal set of gauge theory operators for some examples

In this section, symmetric operators are listed for some examples: $(m,n) = (1,1),(2,1),(3,1)$ and $(2,2)$ using projectors in section 5. As we discussed in sections 4 and 7, symmetric projectors for $k \neq 0$ are labelled by three indices γ, A and i. The index i runs over the multiplicity M^{γ}_{A}. For some examples we will consider here, the multiplicity takes 1 for any γ and A. Therefore we omit the index i in this section. In the case of $k=0$, symmetric projectors are central element of the Brauer algebra and are labelled by an index γ. We express $P^{(k=0,\gamma_+=R,\gamma_-=S)} \equiv P_{R\bar{S}}$.

A.4.1 $m=1$, $n=1$

$$
\begin{aligned}
tr_{1,1}(P_{[1][\bar{1}]}\Phi) &= tr(\Phi)tr(\Phi^\dagger) - \frac{1}{N}tr(\Phi\Phi^\dagger) \\
tr_{1,1}(P^{(k=1)}_{[1][\bar{1}]}\Phi) &= \frac{1}{N}tr(\Phi\Phi^\dagger)
\end{aligned}
\tag{A.37}
$$

To be precise, we should write $P^{(k=1,\gamma_+=\emptyset,\gamma_-=\emptyset)}_{[1][\bar{1}]}$ instead of $P^{(k=1)}_{[1][\bar{1}]}$. But we use the latter expression because there is only one choice for $k=1$ in this case, and this does not cause any confusion. We also use this notation for following examples.

A.4.2 $m=2$, $n=1$

$$
\begin{aligned}
tr_{2,1}(P_{[2][\bar{1}]}\Phi) &= \frac{1}{2}(tr\Phi)^2 tr\Phi^\dagger + \frac{1}{2}tr\Phi^2 tr\Phi^\dagger - tr_{2,1}(P^{(k=1)}_{[2][\bar{1}]}\Phi) \\
tr_{2,1}(P_{[1^2][\bar{1}]}\Phi) &= \frac{1}{2}(tr\Phi)^2 tr\Phi^\dagger - \frac{1}{2}tr\Phi^2 tr\Phi^\dagger - tr_{2,1}(P^{(k=1)}_{[1^2][\bar{1}]}\Phi) \\
tr_{2,1}(P^{(k=1)}_{[2][\bar{1}]}\Phi) &= \frac{1}{N+1}\left(tr(\Phi)tr(\Phi\Phi^\dagger) + tr(\Phi^2\Phi^\dagger)\right) \\
tr_{2,1}(P^{(k=1)}_{[1^2][\bar{1}]}\Phi) &= \frac{1}{N-1}\left(tr(\Phi)tr(\Phi\Phi^\dagger) - tr(\Phi^2\Phi^\dagger)\right)
\end{aligned}
\tag{A.38}
$$

A.4.3 $m=3,\ n=1$

$$
\begin{aligned}
tr_{3,1}(P_{[3][\bar1]}\Phi) &= \frac{1}{6}(tr\Phi)^3 tr\Phi^\dagger + \frac{1}{2}tr\Phi^2 tr\Phi tr\Phi^\dagger + \frac{1}{3}tr\Phi^3 tr\Phi^\dagger - tr_{3,1}(P^{[2][\emptyset](k=1)}_{[3][\bar1]}\Phi)\\
tr_{3,1}(P_{[1^3][\bar1]}\Phi) &= \frac{1}{6}(tr\Phi)^3 tr\Phi^\dagger - \frac{1}{2}tr\Phi^2 tr\Phi tr\Phi^\dagger + \frac{1}{3}tr\Phi^3 tr\Phi^\dagger - tr_{3,1}(P^{[1^2][\emptyset](k=1)}_{[1^3][\bar1]}\Phi)\\
tr_{3,1}(P_{[2,1][\bar1]}\Phi) &= \frac{2}{3}(tr\Phi)^3 tr\Phi^\dagger - \frac{2}{3}tr\Phi^3 tr\Phi^\dagger - tr_{3,1}(P^{[2][\emptyset](k=1)}_{[2,1][\bar1]}\Phi) - tr_{3,1}(P^{[1^2][\emptyset](k=1)}_{[2,1][\bar1]}\Phi)\\
tr_{3,1}(P^{(k=1,\gamma_+=[2],\gamma_-=[\emptyset])}_{[3][\bar1]}\Phi) &= \frac{1}{N+2}\left(\frac{1}{2}tr\Phi\Phi^\dagger(tr\Phi)^2 + \frac{1}{2}tr\Phi\Phi^\dagger tr\Phi^2 + tr\Phi^2\Phi^\dagger tr\Phi + \frac{1}{3}tr\Phi^3\Phi^\dagger\right)\\
tr_{3,1}(P^{(k=1,\gamma_+=[1^2],\gamma_-=[\emptyset])}_{[1^3][\bar1]}\Phi) &= \frac{1}{N-2}\left(\frac{1}{2}tr\Phi\Phi^\dagger(tr\Phi)^2 - \frac{1}{2}tr\Phi\Phi^\dagger tr\Phi^2 - tr\Phi^2\Phi^\dagger tr\Phi + \frac{1}{3}tr\Phi^3\Phi^\dagger\right)\\
tr_{3,1}(P^{(k=1,\gamma_+=[2],\gamma_-=[\emptyset])}_{[2,1][\bar1]}\Phi) &= \frac{1}{N-1}\left(tr\Phi\Phi^\dagger(tr\Phi)^2 + tr\Phi\Phi^\dagger tr\Phi^2 - tr\Phi^3\Phi^\dagger - tr\Phi^2\Phi^\dagger tr\Phi\right)\\
tr_{3,1}(P^{(k=1,\gamma_+=[1^2],\gamma_-=[\emptyset])}_{[2,1][\bar1]}\Phi) &= \frac{1}{N+1}\left(tr\Phi\Phi^\dagger(tr\Phi)^2 - tr\Phi\Phi^\dagger tr\Phi^2 - tr\Phi^3\Phi^\dagger + tr\Phi^2\Phi^\dagger tr\Phi\right)
\end{aligned}
\tag{A.39}
$$

A.4.4 $m=2,\ n=2$

$$
\begin{aligned}
tr_{2,2}(P_{[2][\bar2]}\Phi) &= \frac{1}{4}\left((tr\Phi)^2(tr\Phi^\dagger)^2 + tr\Phi^2(tr\Phi^\dagger)^2 + (tr\Phi)^2 tr(\Phi^\dagger)^2 + tr\Phi^2 tr(\Phi^\dagger)^2\right)\\
&\quad - tr_{2,2}(P^{(k=1)}_{[2][\bar2]}\Phi) - tr_{2,2}(P^{(k=2)}_{[2][\bar2]}\Phi)\\
tr_{2,2}(P_{[2][\bar{1^2}]}\Phi) &= \frac{1}{4}\left((tr\Phi)^2(tr\Phi^\dagger)^2 + tr\Phi^2(tr\Phi^\dagger)^2 - (tr\Phi)^2 tr(\Phi^\dagger)^2 - tr\Phi^2 tr(\Phi^\dagger)^2\right)\\
&\quad - tr_{2,2}(P^{(k=1)}_{[2][\bar{1^2}]}\Phi)\\
tr_{2,2}(P_{[1^2][\bar2]}\Phi) &= \frac{1}{4}\left((tr\Phi)^2(tr\Phi^\dagger)^2 - tr\Phi^2(tr\Phi^\dagger)^2 + (tr\Phi)^2 tr(\Phi^\dagger)^2 - tr\Phi^2 tr(\Phi^\dagger)^2\right)\\
&\quad - tr_{2,2}(P^{(k=1)}_{[1^2][\bar2]}\Phi)\\
tr_{2,2}(P_{[1^2][\bar{1^2}]}\Phi) &= \frac{1}{4}\left((tr\Phi)^2(tr\Phi^\dagger)^2 - tr\Phi^2(tr\Phi^\dagger)^2 - (tr\Phi)^2 tr(\Phi^\dagger)^2 + tr\Phi^2 tr(\Phi^\dagger)^2\right)\\
&\quad - tr_{2,2}(P^{(k=1)}_{[1^2][\bar{1^2}]}\Phi) - tr_{2,2}(P^{(k=2)}_{[1^2][\bar{1^2}]}\Phi)\\
tr_{2,2}(P^{(k=1)}_{[2][\bar2]}\Phi) &= \frac{1}{N+2}\left(tr\Phi\Phi^\dagger tr\Phi tr\Phi^\dagger + tr\Phi^2\Phi^\dagger tr\Phi^\dagger + tr(\Phi(\Phi^\dagger)^2)tr\Phi + tr\Phi^2(\Phi^\dagger)^2\right)\\
&\quad - \frac{2}{N(N+2)}\left((tr\Phi\Phi^\dagger)^2 + tr(\Phi\Phi^\dagger\Phi\Phi^\dagger)\right)\\
tr_{2,2}(P^{(k=1)}_{[2][\bar{1^2}]}\Phi) &= \frac{1}{N}\left(tr\Phi\Phi^\dagger tr\Phi tr\Phi^\dagger + tr\Phi^2\Phi^\dagger tr\Phi^\dagger - tr(\Phi(\Phi^\dagger)^2)tr\Phi - tr\Phi^2(\Phi^\dagger)^2\right)\\
tr_{2,2}(P^{(k=1)}_{[1^2][\bar2]}\Phi) &= \frac{1}{N}\left(tr\Phi\Phi^\dagger tr\Phi tr\Phi^\dagger - tr\Phi^2\Phi^\dagger tr\Phi^\dagger + tr(\Phi(\Phi^\dagger)^2)tr\Phi - tr\Phi^2(\Phi^\dagger)^2\right)\\
tr_{2,2}(P^{(k=1)}_{[1^2][\bar{1^2}]}\Phi) &= \frac{1}{N-2}\left(tr\Phi\Phi^\dagger tr\Phi tr\Phi^\dagger - tr\Phi^2\Phi^\dagger tr\Phi^\dagger - tr(\Phi(\Phi^\dagger)^2)tr\Phi + tr\Phi^2(\Phi^\dagger)^2\right)\\
&\quad - \frac{2}{N(N-2)}\left((tr\Phi\Phi^\dagger)^2 - tr(\Phi\Phi^\dagger\Phi\Phi^\dagger)\right)
\end{aligned}
$$

$$tr_{2,2}(P^{(k=2)}_{[2][\bar{2}]}\Phi) = \frac{1}{N(N+1)}\Big((tr\Phi\Phi^\dagger)^2 + tr(\Phi\Phi^\dagger\Phi\Phi^\dagger)\Big)$$
$$tr_{2,2}(P^{(k=2)}_{[1^2][\bar{1^2}]}\Phi) = \frac{1}{N(N-1)}\Big((tr\Phi\Phi^\dagger)^2 - tr(\Phi\Phi^\dagger\Phi\Phi^\dagger)\Big) \tag{A.40}$$

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